PART - B

Unit - VIII

LINEAR ALGEBRA - 2

8.1 Introduction

In this unit, we continue to discuss a few more matrix oriented concepts such as eigen values and eigen vectors of a square matrix, quadratic form expressible in the matrix form where we see the association with the eigen values and eigen vectors.

8.2 Linear Transformations

Transformation means change.

For example, the reader is familiar with the transformation from cartesian system to polar system. The associated transformation is $x = r\cos\theta$ and $y = r\sin\theta$. Here (x, y) are cartesian coordinates and (r, θ) is expressible in terms of (x, y) as we have, $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}(y/x)$ which being the inverse transformation.

A Linear Transformation in two dimensions is represented by

$$\left.\begin{aligned} y_1 &= a_1 x_1 + a_2 x_2 \\ y_2 &= b_1 x_1 + b_2 x_2 \end{aligned}\right\} \qquad \ldots (1)$$

This can be represented in the matrix form.

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \ldots (2)$$

Similarly a linear transformation in three dimensions along with its matrix form is as follows.

$$\left.\begin{aligned} y_1 &= a_1 x_1 + a_2 x_2 + a_3 x_3 \\ y_2 &= b_1 x_1 + b_2 x_2 + b_3 x_3 \\ y_3 &= c_1 x_1 + c_2 x_2 + c_3 x_3 \end{aligned}\right\} \qquad \ldots (3)$$

or

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \ldots (4)$$

We can as well write (2) and (4) in the form

$$Y = AX \qquad \ldots (5)$$

where Y, A, X are the associated matrices. A is called the ***Transformation Matrix.***

Further if the matrix A is non singular ($|A| \neq 0$) then $Y = AX$ is called a *non singular transformation or regular transformation*. In this case

$$X = A^{-1} Y \qquad \ldots (6)$$

is called the *inverse transformation*.

Also if $|A| = 0$, the transformation $Y = AX$ is called a *singular transformation* .

Next, let $Z = BY$ also be a linear transformation. Then we have,

$$Z = (BY) = B(AX) = (BA)X = CX \ (say) \quad \text{where } C = BA$$

Here $Z = CX$ is called a *composite linear transformation.*

WORKED PROBLEMS

1. *If* $\alpha = x\cos\theta - y\sin\theta$ *and* $\beta = x\sin\theta + y\cos\theta$ *write the matrix* A *of this transformation and prove that* $A^{-1} = A'$. *Also write the inverse transformation.*

>> We have, $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

The matrix $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and $Y = AX$ is the associated matrix representation of the given linear transformation.

$$|A| = \cos^2\theta + \sin^2\theta = 1$$

$$Adj A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} Adj\ A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \qquad \ldots (1)$$

Also $$A' = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \qquad \ldots (2)$$

From (1) and (2), $A^{-1} = A'$

(**Remark** : *This implies that A is an orthogonal matrix*)

The given transformation is $Y = AX$. Hence the associated inverse transformation is,

$$X = A^{-1} Y$$

ie., $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

Thus the inverse transformation is, $x = \alpha\cos\theta + \beta\sin\theta$ **and** $y = -\alpha\sin\theta + \beta\cos\theta$.

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2. *Find the inverse transformation of the following linear transformation.*

$$y_1 = x_1 + 2x_2 + 5x_3$$
$$y_2 = 2x_1 + 4x_2 + 11x_3$$
$$y_3 = -x_2 + 2x_3$$

The given linear transformation in the matrix form is,

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 11 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

ie., $Y = AX$ where we have,

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 11 \\ 0 & -1 & 2 \end{bmatrix}$$

We compute, $A^{-1} = \dfrac{1}{|A|}(Adj\,A)$

$$|A| = 1(8+11) - 2(4-0) + 5(-2-0) = 1$$

$$Adj\,A = \begin{bmatrix} +(8+11), & -(4+5), & +(22-20) \\ -(4-0), & +(2-0), & -(11-10) \\ +(-2-0), & -(-1-0), & +(4-4) \end{bmatrix} = \begin{bmatrix} 19 & -9 & 2 \\ -4 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix}$$

Since $|A| = 1$, $A^{-1} = Adj A$ itself.

Inverse transformation is given by $X = A^{-1}Y$.

ie., $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 19 & -9 & 2 \\ -4 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

Thus, $x_1 = 19y_1 - 9y_2 + 2y_3$, $x_2 = -4y_1 + 2y_2 - y_3$, $x_3 = -2y_1 + y_2$ **is the required inverse transformation.**

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3. Show that the transformation $y_1 = 2x_1 - 2x_2 - x_3$, $y_2 = -4x_1 + 5x_2 + 3x_3$, $y_3 = x_1 - x_2 - x_3$ is regular and find the inverse transformation.

>> The given transformation in the matrix form is,

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -1 \\ -4 & 5 & 3 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

ie., $Y = AX$, where

$$A = \begin{bmatrix} 2 & -2 & -1 \\ -4 & 5 & 3 \\ 1 & -1 & -1 \end{bmatrix}$$

Now, $|A| = 2(-5+3)+2(4-3)-1(4-5) = -1$

$|A| = -1 \neq 0 \Rightarrow$ **the transformation is regular.**

We compute $A^{-1} = \frac{1}{|A|}(Adj\,A)$

$$Adj A = \begin{bmatrix} +(-5+3), & -(2-1), & +(-6+5) \\ -(4-3), & +(-2+1), & -(6-4) \\ +(4-5), & -(-2+2), & +(10-8) \end{bmatrix} = \begin{bmatrix} -2 & -1 & -1 \\ -1 & -1 & -2 \\ -1 & 0 & 2 \end{bmatrix}$$

Hence $A^{-1} = -\begin{bmatrix} -2 & -1 & -1 \\ -1 & -1 & -2 \\ -1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 0 & -2 \end{bmatrix}$

Inverse transformation is given by $X = A^{-1}Y$

ie., $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Thus, $x_1 = 2y_1 + y_2 + y_3$, $x_2 = y_1 + y_2 + 2y_3$, $x_3 = y_1 - 2y_3$ **is the required inverse transformation.**

4. [illegible] $x_1 = 3y_1 + 2y_2$, [illegible] $y_1 = z_1 + 2z_2$ and [illegible] matrices and find the composite transformation [illegible]

>> We have by data,

$$x_1 = 3y_1 + 2y_2, \quad x_2 = -y_1 + 4y_2 \qquad \ldots (1)$$

$$y_1 = z_1 + 2z_2, \quad y_2 = 3z_1 \qquad \ldots (2)$$

The matrix representation of (1) and (2) are as follows.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \text{ and } \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

ie., $X = AY$ and $Y = BZ$

where $A = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$

Now we have, $X = AY = A(BZ) = (AB)Z$

ie., $X = (AB)Z$ is the composite transformation.

Further, $AB = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 3+6, & 6+0 \\ -1+12, & -2+0 \end{bmatrix} = \begin{bmatrix} 9 & 6 \\ 11 & -2 \end{bmatrix}$

$$\therefore \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 9 & 6 \\ 11 & -2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

Thus $\mathbf{x_1 = 9z_1 + 6z_2}$ **&** $\mathbf{x_2 = 11z_1 - 2z_2}$ *is the required composite transformation.*

5. Given the linear transformation [illegible] $y_1 = 5x_1 + 3x_2 + 3x_3$, $y_2 = 3x_1 + 2x_2 - 2x_3$, $y_3 = 2x_1 - x_2 + 2x_3$ and $z_1 = 4x_1 + 2x_3$, $z_2 = x_2 + 4x_3$, $z_3 = 5x_3$, establish the linear transformation [illegible] matrix approach.

>> We have by data,

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 & 3 & 3 \\ 3 & 2 & -2 \\ 2 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \ldots (1)$$

and

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \ldots (2)$$

The equivalent form of (1) and (2) are, $Y = AX$ and $Z = BX$

where $A = \begin{bmatrix} 5 & 3 & 3 \\ 3 & 2 & -2 \\ 2 & -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 0 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

$$Z = BX \Rightarrow X = B^{-1} Z \text{ and hence,}$$

$$Y = AX = A(B^{-1} Z) = (AB^{-1}) Z$$

We need to compute B^{-1} and the matrix prouct AB^{-1}

$$B^{-1} = \frac{1}{|B|} Adj B$$

We have $|B| = 20$

$$Adj\ B = \begin{bmatrix} +(5-0), & -(0-0), & +(0-2) \\ -(0-0), & +(20-0), & -(16-0) \\ +(0-0), & -(0-0), & +(4-0) \end{bmatrix} = \begin{bmatrix} 5 & 0 & -2 \\ 0 & 20 & -16 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\therefore \quad B^{-1} = \frac{1}{20} \begin{bmatrix} 5 & 0 & -2 \\ 0 & 20 & -16 \\ 0 & 0 & 4 \end{bmatrix}$$

Next, $AB^{-1} = \begin{bmatrix} 5 & 3 & 3 \\ 3 & 2 & -2 \\ 2 & -1 & 2 \end{bmatrix} \cdot \frac{1}{20} \begin{bmatrix} 5 & 0 & -2 \\ 0 & 20 & -16 \\ 0 & 0 & 4 \end{bmatrix}$

ie., $AB^{-1} = \frac{1}{20} \begin{bmatrix} 25+0+0, & 0+60+0, & -10-48+12 \\ 15+0+0, & 0+40+0, & -6-32-8 \\ 10+0+0, & 0-20+0, & -4+16+8 \end{bmatrix} = \frac{1}{20} \begin{bmatrix} 25 & 60 & -46 \\ 15 & 40 & -46 \\ 10 & -20 & 20 \end{bmatrix}$

We have $Y = (AB^{-1}) Z$

ie., $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \frac{1}{20} \begin{bmatrix} 25 & 60 & -46 \\ 15 & 40 & -46 \\ 10 & -20 & 20 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$

Thus, $y_1 = (5/4) z_1 + 3 z_2 - (23/10) z_3$

$y_2 = (3/4) z_1 + 2 z_2 - (23/10) z_3$

$y_3 = (1/2) z_1 - z_2 \quad + z_3$ **is the required linear transformation.**

8.3 Eigen values and Eigen vectors of a square matrix

Definition : Given a square matrix A, if there exists a scalar λ (*real or complex*) and a non zero column matrix X such that $AX = \lambda X$, then λ is called an ***eigen value*** of A and X is called an ***eigen vector*** of A corresponding to an eigen value λ.

If I is the unit matrix of the same order as that of A, we have $X = IX$ and hence $AX = \lambda X$ can be written as

$$AX = \lambda(IX) = (\lambda I)X$$

i.e., $[A - \lambda I][X] = [0]$, $[0]$ is the null matrix.

Let us consider a square matrix of order 3 represented by

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \quad \text{Also } \lambda I = \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\therefore \quad [A - \lambda I] = \begin{bmatrix} (a_1 - \lambda) & a_2 & a_3 \\ b_1 & (b_2 - \lambda) & b_3 \\ c_1 & c_2 & (c_3 - \lambda) \end{bmatrix} \quad \text{Also let } X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

It can be easily seen that $[A - \lambda I][X] = [0]$ represents a set of homogeneous equations in 3 unknowns.

i.e.,

$$\begin{aligned} (a_1 - \lambda)x + a_2 y + a_3 z &= 0 \\ b_1 x + (b_2 - \lambda)y + b_3 z &= 0 \\ c_1 x + c_2 y + (c_3 - \lambda)z &= 0 \end{aligned}$$

A nontrivial solution for this system exists if the determinant of the coefficient matrix is zero.

i.e.,

$$\begin{vmatrix} (a_1 - \lambda) & a_2 & a_3 \\ b_1 & (b_2 - \lambda) & b_3 \\ c_1 & c_2 & (c_3 - \lambda) \end{vmatrix} = 0$$

On expanding we get a cubic equation in λ which is called the ***characteristic equation*** of A. The roots of this equation are the *eigen values* which are also called *eigen roots* or *characteristic roots* or *latent roots*. For each value of λ there will be an eigen vector $X \neq 0$ which is also called a *characteristic vector*.

8.31 Properties of eigen values and eigen vectors

1. Sum of the eigen values of a square matrix is equal to the 'trace' (sum of the principal diagonal elements) of the matrix.
2. Product of the eigen values of a square matrix is equal to the determinant of the matrix.
3. If $\lambda_1, \lambda_2, \cdots \lambda_n$ are the eigen values of an n^{th} order square matrix A, then $\lambda_1^k, \lambda_2^k, \lambda_3^k, \cdots \lambda_n^k$ are the eigen values of the matrix A^k.
4. The eigen vector X of a matrix is not unique.
5. If $\lambda_1, \lambda_2, \cdots \lambda_n$ are the distinct eigen values of an n^{th} order square matrix A, then the corresponding eigen vectors $X_1, X_2, \cdots, X_n$ form a linearly independent set.
6. If two or more eigen values are equal it may or may not be possible to get linearly independent eigen vectors corresponding to the coincident eigen values.
7. If $X_1 = (x_1, y_1, z_1)$, $X_2 = (x_2, y_2, z_2)$ then X_1, X_2 are called orthogonal vectors if,

   $$X_1 \cdot X_2 = x_1 x_2 + y_1 y_2 + z_1 z_2 = 0$$
8. Eigen vectors of a symmetric matrix corresponding to different eigen values are orthogonal.
9. 'Norm' of a vector $X = (x, y, z)$ denoted by $|| \times ||$ is equal to $\sqrt{x^2 + y^2 + z^2} = k\,(say)$. Then $\left(\frac{x}{k}, \frac{y}{k}, \frac{z}{k}\right)$ is called the normalized vector.
10. The matrix $P = \left[\frac{X_1'}{|| X_1 ||}, \frac{X_2'}{|| X_2 ||}, \frac{X_3'}{|| X_3 ||}\right]$ will be an orthogonal matrix.

Working procedure for problems

- Given a square matrix A (*usually of order* 3) we form $| A - \lambda I | = 0$.
  On expanding we get the characteristic equation of A. By solving it we get all the eigen values.
- We then form the system of homogeneous equations from the matrix equation $[A - \lambda I]\,[X] = [0]$ and solve for (x, y, z) corresponding to every value of λ.
- Simple techniques of solving or the rule of cross multiplication (*for any pair of equations*) can be employed. The values x, y, z obtained by the rule of cross multiplication satisfy simultaneously all the three equations.

WORKED PROBLEMS

6. *Find all the eigen values and the corresponding eigen vectors of the matrix*

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

>> The characteristic equation of A is $|A - \lambda I| = 0$

i.e.,
$$\begin{vmatrix} (8-\lambda) & -6 & 2 \\ -6 & (7-\lambda) & -4 \\ 2 & -4 & (3-\lambda) \end{vmatrix} = 0$$

On expanding we have,

$$(8-\lambda)\,[(7-\lambda)(3-\lambda)-16] + 6\,[-6(3-\lambda)+8] + 2\,[24-2(7-\lambda)] = 0$$

i.e., $(8-\lambda)\,[5-10\lambda+\lambda^2] + 6\,[6\lambda-10] + 2\,[10+2\lambda] = 0$

i.e., $-\lambda^3 + 18\lambda^2 - 45\lambda = 0$, on simplification.

or $\lambda^3 - 18\lambda^2 + 45\lambda = 0$

i.e., $\lambda\,(\lambda^2 - 18\lambda + 45) = 0$ or $\lambda(\lambda-3)(\lambda-15) = 0$

$\therefore$ $\lambda = 0, 3, 15$ are the eigen values of A.

We now form the system of equations

$$\begin{aligned} (8-\lambda)x - 6y + 2z &= 0 \\ -6x + (7-\lambda)y - 4z &= 0 \\ 2x - 4y + (3-\lambda)z &= 0 \end{aligned} \qquad \ldots (1)$$

Case - i : Let $\lambda = 0$. The system of equations become

$8x - 6y + 2z = 0$ ... (i)

$-6x + 7y - 4z = 0$ ... (ii)

$2x - 4y + 3z = 0$ ... (iii)

Applying the rule of cross multiplication for (i) and (ii)

$$\frac{x}{\begin{vmatrix} -6 & 2 \\ 7 & -4 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 8 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}}$$

i.e., $\dfrac{x}{10} = \dfrac{y}{20} = \dfrac{z}{20}$ or $\dfrac{x}{1} = \dfrac{y}{2} = \dfrac{z}{2}$

$\therefore$ (x, y, z) are proportional to $(1, 2, 2)$ and we can write $x = k, y = 2k, z = 2k$ where k is arbitrary. However it is enough to keep the values of (x, y, z) in the simplest form $x = 1, y = 2, z = 2$. These values satisfy all the equations simultaneously.

Thus the eigen vector X_1 corresponding to the eigen value $\lambda = 0$ is $X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

[*The same will be written as a row vector in the form* $X_1 = (1, 2, 2)$ *in future*]

Thus $X_1 = (1, 2, 2)$ is the eigen vector corresponding to $\lambda = 0$.

Case - ii : Let $\lambda = 3$ and the corresponding equations from (1) are

$$5x - 6y + 2z = 0 \qquad \ldots \text{(iv)}$$

$$-6x + 4y - 4z = 0 \qquad \ldots \text{(v)}$$

$$2x - 4y + 0z = 0 \qquad \ldots \text{(vi)}$$

From (iv) and (v) we have as before,

$$\frac{x}{24 - 8} = \frac{-y}{-20 + 12} = \frac{z}{20 - 36} \quad \text{or} \quad \frac{x}{16} = \frac{-y}{-8} = \frac{z}{-16}$$

i.e., $\frac{x}{2} = \frac{y}{1} = \frac{z}{-2}$ $\quad \therefore \quad (x, y, z) = (2, 1, -2)$

Thus $\quad X_2 = (2, 1, -2)$ is the eigen vector corresponding to $\lambda = 3$.

Case - iii : Let $\lambda = 15$ and the associated equations from (1) are

$$-7x - 6y + 2z = 0 \qquad \ldots \text{(vii)}$$

$$-6x - 8y - 4z = 0 \qquad \ldots \text{(viii)}$$

$$2x - 4y - 12z = 0 \qquad \ldots \text{(ix)}$$

From (vii) and (viii) we have,

$$\frac{x}{24 + 16} = \frac{-y}{28 + 12} = \frac{z}{56 - 36} \quad \text{or} \quad \frac{x}{40} = \frac{-y}{40} = \frac{z}{20}$$

i.e., $\frac{x}{2} = \frac{y}{-2} = \frac{z}{1}$ $\quad \therefore \quad (x, y, z) = (2, -2, 1)$

Thus $\quad X_3 = (2, -2, 1)$ is the eigen vector corresponding to $\lambda = 15$.

Note : *The characteristic equation of a third order square matrix A can be obtained without expanding* $|A - \lambda I| = 0$ *by the following rule :*

$$\lambda^3 - (\Sigma d)\lambda^2 + (\Sigma m_d)\lambda - |A| = 0, \text{ where}$$

Σd = Sum of the diagonal elements of A

Σm_d = Sum of the minors of the diagonal elements of A

$|A|$ = Determinant of A

In the Example - 6

$$\Sigma d = 8 + 7 + 3 = 18$$

$$\Sigma m_d = \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix} = 5 + 20 + 20 = 45$$

$$|A| = 8(21 - 16) + 6(-18 + 8) + 2(24 - 14) = 40 - 60 + 20 = 0$$

Substituting in the rule we get the characteristic equation,

$$\lambda^3 - 18\lambda^2 + 45\lambda = 0$$

Remark : *Observe the verification of some of the properties connected with eigen values and eigen vectors stated earlier.*

(i) Sum of all the eigen values = $0 + 3 + 15 = 18$ is equal to the trace of A which being $8 + 7 + 3 = 18$

(ii) Product of all the eigen values $= 0$. $|A| = 0$

(iii) $X_1 \cdot X_2 = 2 + 2 - 4 = 0$, $X_2 \cdot X_3 = 4 - 2 - 2 = 0$, $X_3 \cdot X_1 = 2 - 4 + 2 = 0$
$\Rightarrow$ Eigen vectors X_1, X_2, X_3 are orthogonal.

(iv) $\|X_1\| = \sqrt{1+4+4} = 3$, $\|X_2\| = \sqrt{4+1+4} = 3$, $\|X_3\| = \sqrt{4+4+1} = 3$
Normalized eigen vectors of X_1, X_2, X_3 are respectively $(1/3, 2/3, 2/3)$, $(2/3, 1/3, -2/3)$, $(2/3, -2/3, 1/3)$

The matrix $P = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ is an orthogonal matrix.

($PP' = I$ *can be easily verified*).

7 *Find all the eigen values and the corresponding eigen vectors for the matrix*

$$A = \begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{bmatrix}$$

>> $|A - \lambda I| = 0$ is the characteristic equation of A .

i.e., $\begin{vmatrix} (7-\lambda) & -2 & 0 \\ -2 & (6-\lambda) & -2 \\ 0 & -2 & (5-\lambda) \end{vmatrix} = 0$

or $(7-\lambda)[(6-\lambda)(5-\lambda)-4]+2[-2(5-\lambda)] = 0$

i.e., $-\lambda^3 + 18\lambda^2 - 99\lambda + 162 = 0$, on simplification.

or $\lambda^3 - 18\lambda^2 + 99\lambda - 162 = 0$

To solve this cubic equation we shall first find a root by inspection by simply trying values for $\lambda = 1, 2, 3, \ldots$ *(If λ is negative all the terms of the equation will be negative and hence cannot become zero)*

Putting $\lambda = 3$ we have $27 - 162 + 297 - 162 = 324 - 324 = 0$

Thus $\lambda = 3$ is a root by inspection. The other two roots can be found by synthetic division as follows.

3	1	− 18	99	− 162
	0	+ 3	− 45	+ 162
	1	− 15	+ 54	0

$\therefore$ the quadratic is $\lambda^2 - 15\lambda + 54 = 0$

i.e., $(\lambda-6)(\lambda-9) = 0$ or $\lambda = 6, 9$

Thus $\boldsymbol{\lambda = 3, 6, 9}$ are the eigen values.

We now form the system of equations

$$\begin{aligned} (7-\lambda)x \quad -2y \quad\quad +0z &= 0 \\ -2x \quad +(6-\lambda)y - 2z &= 0 \\ 0x \quad -2y \quad +(5-\lambda)z &= 0 \end{aligned} \qquad \ldots(1)$$

Case - i : Let $\lambda = 3$ and the corresponding equations are

$4x - 2y + 0z = 0$ ...(i)

$-2x + 3y - 2z = 0$ ...(ii)

$0x - 2y + 2z = 0$ ...(iii)

From (i) and (ii) we have by applying the rule of cross multiplication,

$$\frac{x}{4-0} = \frac{-y}{-8-0} = \frac{z}{12-4} \text{ or } \frac{x}{4} = \frac{y}{8} = \frac{z}{8} \text{ or } \frac{x}{1} = \frac{y}{2} = \frac{z}{2}$$

$\therefore$ $X_1 = (1, 2, 2)$ is the eigen vector corresponding to $\lambda = 3$.

Case - ii : Let $\lambda = 6$ and the corresponding equations from (1) are

$$1x - 2y + 0z = 0 \quad \ldots \text{(iv)}$$
$$-2x - 0y - 2z = 0 \quad \ldots \text{(v)}$$
$$0x - 2y - 1z = 0 \quad \ldots \text{(vi)}$$

From (iv) and (v), $\frac{x}{4} = \frac{-y}{-2} = \frac{z}{-4}$ or $\frac{x}{2} = \frac{y}{1} = \frac{z}{-2}$

$\therefore$ $X_2 = (2, 1, -2)$ is the eigen vector corresponding to $\lambda = 6$.

Case - iii : Let $\lambda = 9$ and the corresponding equations from (1) are

$$-2x - 2y + 0z = 0 \quad \ldots \text{(vii)}$$
$$-2x - 3y - 2z = 0 \quad \ldots \text{(viii)}$$
$$0x - 2y - 4z = 0 \quad \ldots \text{(ix)}$$

From (vii) and (viii), $\frac{x}{4} = \frac{-y}{4} = \frac{z}{2}$ or $\frac{x}{2} = \frac{y}{-2} = \frac{z}{1}$

$\therefore$ $X_3 = (2, -2, 1)$ is the eigen vector corresponding to $\lambda = 9$.

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8. *Find the characteristic roots and the corresponding characteristic vectors for the following matrix*

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

>> The characteristic equation of the given matrix is

$$\begin{vmatrix} (2-\lambda) & 0 & 1 \\ 0 & (2-\lambda) & 0 \\ 1 & 0 & (2-\lambda) \end{vmatrix} = 0$$

i.e., $(2-\lambda)(2-\lambda)^2 - (2-\lambda) = 0$

i.e., $(2-\lambda)\left[(2-\lambda)^2 - 1\right] = 0$

or $(2-\lambda)(2-\lambda+1)(2-\lambda-1) = 0$

i.e., $(2-\lambda)(3-\lambda)(1-\lambda) = 0$ or $\lambda = 2, \lambda = 3, \lambda = 1$

Thus $\lambda = 1, 2, 3$ are the characteristic roots.

Let us now form the system of equations

$$\begin{aligned} (2-\lambda)x &+ 0y &+ 1z &= 0 \\ 0x &+ (2-\lambda)y &+ 0z &= 0 \\ x &+ 0y &+ (2-\lambda)z &= 0 \end{aligned} \qquad \ldots (1)$$

Case - i : Let $\lambda = 1$ and the corresponding equations are

$$x + z = 0, \quad y = 0, \quad x + z = 0$$

i.e., $x = -z$ and if $z = 1$ is arbitrarily chosen for convenience then $x = -1$

$\therefore \ (x, y, z) = (-1, 0, 1)$ [*The rule of cross multiplication is not used as the equations are highly simple*]

$\therefore \quad \mathbf{X_1 = (-1, 0, 1)}$ is the eigen vector corresponding to $\lambda = 1$.

Case - ii : Let $\lambda = 2$ and the corresponding equations from (1) are $z = 0,\ 0 = 0,\ x = 0$ since $x = 0$ and $z = 0$, y can be chosen arbitrarily, say $y = 1$.

$\therefore \quad \mathbf{X_2 = (0, 1, 0)}$ is the eigen vector corresponding to $\lambda = 2$.

Case - iii : Let $\lambda = 3$ and the corresponding equations from (1) are $-x + z = 0$, $-y = 0,\ x - z = 0 \ \therefore\ x = z$ and $y = 0$

Let us choose $x = z = 1$ *(arbitrary)*

$\therefore \quad \mathbf{X_3 = (1, 0, 1)}$ is the eigen vector corresponding to $\lambda = 3$.

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9. *Find the eigen roots and the corresponding eigen vectors for the matrix*

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

\>> $\quad |A - \lambda I| = 0$ is the characteristic equation of A.

i.e., $\begin{vmatrix} (-2-\lambda) & 2 & -3 \\ 2 & (1-\lambda) & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$

i.e., $(-2-\lambda)[-\lambda(1-\lambda) - 12] - 2[-2\lambda - 6] - 3[-4 + 1 - \lambda] = 0$

i.e., $(-2-\lambda)(-\lambda + \lambda^2 - 12) + (4\lambda + 12) + (9 + 3\lambda) = 0$

i.e., $(-2-\lambda)(\lambda + 3)(\lambda - 4) + 4(\lambda + 3) + 3(\lambda + 3) = 0$

i.e., $(\lambda + 3)[(-2-\lambda)(\lambda - 4) + 4 + 3] = 0$

i.e., $(\lambda + 3)(-\lambda^2 + 2\lambda + 15) = 0$ or $(\lambda + 3)(\lambda^2 - 2\lambda - 15) = 0$

i.e., $(\lambda + 3)(\lambda + 3)(\lambda - 5) = 0$ or $\lambda = -3, -3, 5$

$\therefore \quad \lambda_1 = -3, \ \lambda_2 = -3, \ \lambda_3 = 5$ are the eigen values.

We now form the system of equations.

$$\begin{aligned} (-2-\lambda)x + 2y \quad & -3z = 0 \\ 2x + (1-\lambda)y & - 6z = 0 \\ -1x - 2y \quad & -\lambda z = 0 \end{aligned} \qquad \ldots (1)$$

Case - i : Let $\lambda = -3$ and the corresponding equations are

$$x + 2y - 3z = 0 \qquad \ldots (i)$$

$$2x + 4y - 6z = 0 \qquad \ldots (ii)$$

$$-x + 2y + 3z = 0 \qquad \ldots (iii)$$

It should be observed that the equations (i), (ii), (iii) are all same and we have only one independent equation $x + 2y - 3z = 0$ (*In case the rule of cross multiplication is applied, we get* $x = 0 = y = z$ *which is a trivial solution*)

Two variables can be chosen arbitrarily.

Let $\quad z = k_1,\ y = k_2 \ \therefore\ x = 3k_1 - 2k_2$

Thus $\quad X_1 = (3k_1 - 2k_2,\ k_2,\ k_1)$ is the eigen vector corresponding to $\lambda = -3$. where $k_1,\ k_2$ are not simultaneously zero.

Case - ii : Let $\lambda = 5$ and the corresponding equations from (1) are

$$-7x + 2y - 3z = 0 \qquad \ldots (iv)$$

$$2x - 4y - 6z = 0 \qquad \ldots (v)$$

$$-1x - 2y - 5z = 0 \qquad \ldots (vi)$$

From (iv) and (v), $\dfrac{x}{-12-12} = \dfrac{-y}{42+6} = \dfrac{z}{28-4}$

i.e., $\quad \dfrac{x}{-24} = \dfrac{-y}{48} = \dfrac{z}{24} \quad$ or $\quad \dfrac{x}{1} = \dfrac{y}{2} = \dfrac{z}{-1}$

$\therefore \quad X_2 = (1,\ 2,\ -1)$ is the eigen vector corresponding to $\lambda = 5$.

---

10. *Find the eigen values and eigen vectors of the matrix*

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

>> The eigen values are obtained from the characteristic equation $|A - \lambda I| = 0$.

ie., $$\begin{vmatrix} (1-\lambda) & 1 & 3 \\ 1 & (5-\lambda) & 1 \\ 3 & 1 & (1-\lambda) \end{vmatrix} = 0$$

On expanding we get $\lambda^3 - 7\lambda^2 + 36 = 0$.

$\lambda = -2$ is a root by inspection. Now by synthetic division,

$$\begin{array}{r|rrrr} -2 & 1 & -7 & 0 & 36 \\ & 0 & -2 & 18 & -36 \\ \hline & 1 & -9 & 18 & 0 \end{array}$$

$\Rightarrow$ $\lambda^2 - 9\lambda + 18 = 0$ or $(\lambda - 3)(\lambda - 6) = 0$ or $\lambda = 3, \lambda = 6$

$\therefore$ $\lambda = -2, 3, 6$ are the eigen values.

We now form the system of equations,

$$\left.\begin{aligned} (1-\lambda)x + 1y + 3z &= 0 \\ 1x + (5-\lambda)y + 1z &= 0 \\ 3x + 1y + (1-\lambda)z &= 0 \end{aligned}\right\} \quad \ldots (1)$$

Case (i) : Let $\lambda = -2$ and the corresponding equations are

$$\left.\begin{aligned} 3x + 1y + 3z &= 0 \\ 1x + 7y + z &= 0 \\ 3x + 1y + 3z &= 0 \end{aligned}\right\} \Rightarrow \frac{x}{-20} = \frac{-y}{0} = \frac{z}{20}$$

$\therefore$ $(x, y, z) = (1, 0, -1)$ is the eigen vector corresponding to $\lambda = -2$.

Case (ii) : Let $\lambda = 3$ and we have from (1),

$$\left.\begin{aligned} -2x + 1y + 3z &= 0 \\ 1x + 2y + 1z &= 0 \\ 3x + 1y - 2z &= 0 \end{aligned}\right\} \Rightarrow \frac{x}{-5} = -\frac{-y}{-5} = \frac{z}{-5}$$

$\therefore$ $(x, y, z) = (1, -1, 1)$ is the eigen vector corresponding to $\lambda = 3$.

Case (iii) : Let $\lambda = 6$ and we have from (1),

$$\left.\begin{aligned} -5x + 1y + 3z &= 0 \\ 1x - 1y + 1z &= 0 \\ 3x + 1y - 5z &= 0 \end{aligned}\right\} \Rightarrow \frac{x}{4} = -\frac{-y}{-8} = \frac{z}{4} \text{ or } \frac{x}{1} = \frac{y}{2} = \frac{z}{1}$$

$\therefore$ $(x, y, z) = (1, 2, 1)$ is the eigen vector corresponding to $\lambda = 6$.

Thus $-2, 3, 6$ are the eigen values and the corresponding eigen vectors are $(1, 0, -1)$; $(1, -1, 1)$; $(1, 2, 1)$

>> $| A - \lambda I | = 0$ is the characteristic equation of A.

i.e.,
$$\begin{vmatrix} (6-\lambda) & -2 & 2 \\ -2 & (3-\lambda) & -1 \\ 2 & -1 & (3-\lambda) \end{vmatrix} = 0$$

On expanding, we obtain

$$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

$\lambda = 2$ is a root by inspection. Now by synthetic division,

2	1	−12	36	−32
	0	+2	−20	+32
	1	−10	16	: 0

Now by solving $\lambda^2 - 10\lambda + 16 = 0$ we obtain $(\lambda - 2)(\lambda - 8) = 0$

$\therefore \quad \lambda = 2, \ 8$

Thus **$\lambda = 2, \ 2, \ 8$ are the eigen values.**

We now form the equations,

$$\begin{aligned} (6-\lambda)x \quad -2y \quad\quad +2z \quad &= 0 \\ -2x \quad +(3-\lambda)y \quad -1z \quad &= 0 \\ 2x \quad -1y \quad +(3-\lambda)z &= 0 \end{aligned} \quad \ldots (1)$$

Case - (i) : Let $\lambda = 2$ and the corresponding equations are,

$4x - 2y + 2z = 0$ ... (i)

$-2x + y - z = 0$ ... (ii)

$2x - y + z = 0$ ... (iii)

The above set of equations are all same as we have only one independent equation $2x - y + z = 0$ and hence we can choose two variables arbitrarily.

Let $z = k_1$ and $y = k_2 \ \therefore \ x = (k_2 - k_1)/2$

$\therefore \ X_1 = \left[(k_2 - k_1)/2, \ k_2, \ k_1 \right]$ is the eigen vector corresponding to $\lambda = 2$ where $k_1, \ k_2$ are not simultaneously equal to zero.

Case - (ii) Let $\lambda = 8$ and we have from (1)

$$\left.\begin{aligned} -2x-2y+2z &= 0 \\ -2x-5y-1z &= 0 \\ 2x-1y-5z &= 0 \end{aligned}\right\} \Rightarrow \frac{x}{12} = \frac{-y}{6} = \frac{z}{6} \quad \text{or} \quad \frac{x}{2} = \frac{y}{-1} = \frac{z}{1}$$

$\therefore$ $X_2 = (2, -1, 1)$ **is the eigen vector corresponding to $\lambda = 8$.**

>> The charecteristic equation of A is $|A - \lambda I| = 0$.

ie.,
$$\begin{vmatrix} (-3-\lambda) & -7 & -5 \\ 2 & (4-\lambda) & 3 \\ 1 & 2 & (2-\lambda) \end{vmatrix} = 0$$

ie., $(-3-\lambda)\ [8-6\lambda+\lambda^2-6]+7\ [4-2\lambda-3]-5\ [4-4+\lambda] = 0$

ie., $(-3-\lambda)\ (\lambda^2-6\lambda+2)+7(1-2\lambda)-5(\lambda) = 0$

$-3\lambda^2+18\lambda-6-\lambda^3+6\lambda^2-2\lambda+7-14\lambda-5\lambda = 0$

ie., $-\lambda^3+3\lambda^2-3\lambda+1 = 0$

or $\lambda^3-3\lambda^2+3\lambda-1 = 0$

or $(\lambda-1)^3 = 0 \Rightarrow \lambda = 1, 1, 1$. **All the eigen values are equal.**

We now form the system of equations

$$\begin{aligned} (-3-\lambda)x - 7y - 5z &= 0 \\ 2x + (4-\lambda)y + 3z &= 0 \\ 1x + 2y + (2-\lambda)z &= 0 \end{aligned}$$

Putting $\lambda = 1$ we obtain,

$$\left.\begin{aligned} -4x-7y-5z &= 0 \\ 2x+3y+3z &= 0 \\ 1x+2y+1z &= 0 \end{aligned}\right\} \Rightarrow \frac{x}{-6} = \frac{-y}{-2} = \frac{z}{2} \quad \text{or} \quad \frac{x}{3} = \frac{y}{-1} = \frac{z}{-1}$$

Thus $X = (3, -1, -1)$, **is the eigen vector corresponding to the coincident eigen value $\lambda = 1$.**

>> Sum of all the eigen values of the given matrix is equal to the 'trace' of the given matrix.

ie., $= 2+3+2 = 7$ (*Sum of the principal diagonal elements*)

Next, product of all the eigen values is equal to the determinant of the given matrix.

ie., $\begin{vmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{vmatrix} = 2(6-2) - 2(2-1) + 1(2-3) = 8-2-1 = \mathbf{5}$

Thus the sum and product of all the eigen values of the given matrix are respectively 7 and 5.

Two square matrices A and B of the same order are said to be **similar** if there exists a non singular matrix P such that

$$B = P^{-1} A P$$

Here B is said to be similar to A .

Diagonalisation of a square matrix

Property : *If A is a square matrix of order n having n lineraly independent eigen vectors then there exists an* n^{th} *order square matrix P such that* $P^{-1} A P$ *is a diagonal matrix.*

We shall establish this result by considering a third order square matrix to make an important and interesting observation.

Let A be a third order square matrix having eigen values $\lambda_1, \lambda_2, \lambda_3$ and the corresponding eigen vectors.

$$X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \text{ and } X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$$

Let the square matrix P be equal to $[\, X_1 \ X_2 \ X_3 \,]$.

ie., $P = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$

Now $AP = A[X_1 \; X_2 \; X_3] = [AX_1 \; AX_2 \; AX_3] = [\lambda_1 X_1, \lambda_2 X_2, \lambda_3 X_3]$

or $AP = \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \lambda_3 x_3 \\ \lambda_1 y_1 & \lambda_2 y_2 & \lambda_3 y_3 \\ \lambda_1 z_1 & \lambda_2 z_2 & \lambda_3 z_3 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

ie., $AP = PD$ where D is the diagonal matrix represnted by

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Consider $AP = PD$

Pre multiplying by P^{-1} we have,

$$P^{-1}AP = P^{-1}PD = (P^{-1}P)D = ID = D$$

Thus $\mathbf{P^{-1}AP = D}$

It is important to note that $P^{-1}AP$ is a diagonal matrix having the eigen values of A, $(\lambda_1, \lambda_2, \lambda_3)$ in its principal diagonal. We say that the matrix P diagonalizes A where P is constituted by the eigen vectors of A.

Note : (1) *The transformation of a square matrix A to $P^{-1}AP$ is known as **Similarity Transformation.***

(2) *The matrix P which diagonalizes A is called the modal matrix of A and the resulting diagonal matrix is called the **spectral matrix** of A.*

Computation of powers of a square matrix

Diagonalization of a square matrix A also helps us to find the powers of A: $A^2, A^3, A^4, \cdots$ etc.,

We have $D = P^{-1}AP$

$\therefore \quad D^2 = (P^{-1}AP)(P^{-1}AP) = P^{-1}A(PP^{-1})AP = P^{-1}AIAP = P^{-1}A^2P$

ie., $D^2 = P^{-1}A^2P$

Pre multiplying by P and post multiplying by P^{-1} we have,

$$P\,D^2\,P^{-1} = (PP^{-1})\,A^2\,(PP^{-1}) = I A^2 I = A^2$$

ie., $\quad A^2 = P\,D^2\,P^{-1}$

Thus in general, $A^n = P D^n P^{-1}$, where

$$D^n = \begin{bmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{bmatrix}$$

Working procedure for diagonalization of a square matrix A of order 3

- We find eigen values $\lambda_1, \lambda_2, \lambda_3$
- We find the eigen vectors X_1, X_2, X_3 corresponding to the eigen values $\lambda_1, \lambda_2, \lambda_3$
- We form the modal matrix $P = [\,X_1 \;\; X_2 \;\; X_3\,] = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$
- We compute $P^{-1} = \dfrac{1}{|P|}\,(Adj\;P)$
- Finally we compute $P^{-1} A P$

The diagonalization of A is given by $D = P^{-1} A P$

where we obtain $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

WORKED PROBLEMS

14. *Reduce the matrix* $A = \begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix}$ *to the diagonal form and hence find* A^4.

>> The characteristic equation of A is $|A - \lambda I| = 0$.

ie., $\quad \begin{vmatrix} (-1-\lambda) & 3 \\ -2 & (4-\lambda) \end{vmatrix} = 0$

ie., $\quad (-1-\lambda)\,(4-\lambda) + 6 = 0$

ie., $\quad \lambda^2 - 3\lambda + 2 = 0$

or $\quad (\lambda - 1)\,(\lambda - 2) = 0 \quad \therefore \; \lambda = 1$ and 2 are the eigen values of A.

Now consider $[A - \lambda I]\ [X] = [0]$

ie., $\begin{bmatrix} (-1-\lambda) & 3 \\ -2 & (4-\lambda) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

ie., $(-1-\lambda)\, x + 3y = 0$

$-2x + (4-\lambda)\, y = 0$

Case - (i) : Let $\lambda = 1$

We get $-2x + 3y = 0$ or $2x = 3y$ or $\frac{x}{3} = \frac{y}{2}$

$\therefore \quad X_1 = (3, 2)'$ is the eigen vector corresponding to $\lambda = 1$.

Case - (ii) : Let $\lambda = 2$

We get $-3x + 3y = 0$ or $x = y$ or $\frac{x}{1} = \frac{y}{1}$

$\therefore \quad X_2 = (1, 1)'$ is the eigen vector corresponding to $\lambda = 2$.

Modal matrix $P = [X_1\ X_2] = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$

We have $|P| = 1$ and $P^{-1} = \frac{1}{|P|} (Adj P)$

$$P^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

Now $P^{-1} A P = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

Thus $P^{-1} A P = D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ is the diagonal matrix.

or $\quad P^{-1} A P = Diag(1, 2)$

Also we have $A^n = P D^n P^{-1}$

$\therefore \quad A^4 = P D^4 P^{-1}$ where $D^4 = \begin{bmatrix} 1^4 & 0 \\ 0 & 2^4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 16 \end{bmatrix}$

ie., $A^4 = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 16 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -32 & 48 \end{bmatrix} = \begin{bmatrix} -29 & 45 \\ -30 & 46 \end{bmatrix}$

Thus $A^4 = \begin{bmatrix} -29 & 45 \\ -30 & 46 \end{bmatrix}$

>> Let $A = \begin{bmatrix} -19 & 7 \\ -42 & 16 \end{bmatrix}$

The characteristic equation of A is $| A - \lambda I | = 0$.

ie., $\begin{vmatrix} (-19-\lambda) & 7 \\ -42 & (16-\lambda) \end{vmatrix} = 0$

ie., $\lambda^2 + 3\lambda - 304 + 294 = 0$

ie., $\lambda^2 + 3\lambda - 10 = 0$

or $(\lambda - 2)(\lambda + 5) = 0$

ie., $\lambda = 2, -5$ are the eigen values of A.

Now consider, $[A - \lambda I][X] = [0]$

ie., $\begin{bmatrix} (19-\lambda) & 7 \\ -42 & (16-\lambda) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

ie., $(-19-\lambda)x + 7y = 0$

$-42x + (16-\lambda)y = 0$

Case-(i) : Let $\lambda = 2$

We get $-21x + 7y = 0$ and $-42x + 14y = 0$

ie., $y = 3x$ or $\frac{y}{3} = \frac{x}{1}$

$\therefore$ $X_1 = (1, 3)'$ is the eigen vector corresponding to $\lambda = 2$.

Case-(ii) : Let $\lambda = -5$

We get $-14x + 7y = 0$ and $-42x + 21y = 0$

ie., $y = 2x$ or $\frac{y}{2} = \frac{x}{1}$

$\therefore$ $X_2 = (1, 2)'$ is the eigen vector corresponding to $\lambda = -5$.

Modal matrix $P = [x_1 \, x_2] = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}$

We have $| P | = 2 - 3 = -1$ and $P^{-1} = \frac{1}{|P|}(\text{Adj } P)$

$$P^{-1} = -\begin{bmatrix} 2 & -1 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix}$$

Now $P^{-1} AP = D = \begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} -19 & 7 \\ -42 & 16 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ -15 & 5 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}$

Thus $P^{-1}AP = D = \begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}$ **is the diagonal matrix.**

or $P^{-1}AP = Diag\ (2, -5)$

>> The characteristic equation of A is $| A - \lambda I | = 0$

ie., $$\begin{vmatrix} (11-\lambda) & -4 & -7 \\ 7 & (-2-\lambda) & -5 \\ 10 & -4 & (-6-\lambda) \end{vmatrix} = 0$$

ie., $(11-\lambda)[(-2-\lambda)(-6-\lambda)-20] + 4[7(-6-\lambda)+50$

$-7[-28-10(-2-\lambda)] = 0$

ie., $(11-\lambda)[\lambda^2+8\lambda-8]+4[8-7\lambda]-7[10\lambda-8] = 0$

ie., $11\lambda^2+88\lambda-88-\lambda^3-8\lambda^2+8\lambda+32-28\lambda-70\lambda+56 = 0$

ie., $-\lambda^3+3\lambda^2-2\lambda = 0$ or $\lambda^3-3\lambda^2-2\lambda = 0$ or $\lambda(\lambda^2-3\lambda+2) = 0$

or $\lambda(\lambda-1)(\lambda-2) = 0 \Rightarrow \lambda = 0, 1, 2$

Now consider $[A-\lambda I][X] = [0]$

ie., $(11-\lambda)x - 4y - 7z = 0$

$7x + (-2-\lambda)y - 5z = 0$

$10x - 4y + (-6-\lambda)z = 0$

Case - (i) : Let $\lambda = 0$ and the corresponding equations are

$$\left.\begin{aligned} 11x-4y-7z &= 0 \\ 7x-2y-5z &= 0 \\ 10x-4y-6z &= 0 \end{aligned}\right\} \Rightarrow \frac{x}{6} = \frac{-y}{-6} = \frac{z}{6} \text{ or } \frac{x}{1} = \frac{y}{1} = \frac{z}{1}$$

$X_1 = (1, 1, 1)'$ is the eigen vector corresponding to $\lambda = 0$.

Case (ii) : Let $\lambda = 1$ and the corresponding equations are

$$\left.\begin{array}{r} 10x - 4y - 7z = 0 \\ 7x - 3y - 5z = 0 \\ 10x - 4y - 7z = 0 \end{array}\right\} \Rightarrow \frac{x}{-1} = \frac{-y}{-1} = \frac{z}{-2} \text{ or } \frac{x}{1} = \frac{y}{-1} = \frac{z}{2}$$

$X_2 = (1, -1, 2)'$ is the eigen vector corresponding to $\lambda = 1$.

Case-(iii) : Let $\lambda = 2$ and the corresponding equations are

$$\left.\begin{array}{r} 9x - 4y - 7z = 0 \\ 7x - 4y - 5z = 0 \\ 10x - 4y - 8z = 0 \end{array}\right\} \Rightarrow \frac{x}{-8} = \frac{-y}{4} = \frac{z}{-8} \text{ or } \frac{x}{2} = \frac{y}{1} = \frac{z}{2}$$

$X_3 = (2, 1, 2)'$ is the eigen vector corresponding to $\lambda = 2$.

Hence the modal matrix $P = [X_1 \; X_2 \; X_3] = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

We have $\quad |P| = 1(-2-2) - 1(2-1) + 2(2+1) = 1$

$$Adj\ P = \begin{bmatrix} +(-2-2), & -(2-4), & +(1+2) \\ -(2-1), & +(2-2), & -(1-2) \\ +(2+1), & -(2-1), & +(-1-1) \end{bmatrix} = \begin{bmatrix} -4 & 2 & 3 \\ -1 & 0 & 1 \\ 3 & -1 & -2 \end{bmatrix}$$

$$P^{-1} = \frac{1}{|P|}(Adj\,P) = \begin{bmatrix} -4 & 2 & 3 \\ -1 & 0 & 1 \\ 3 & -1 & -2 \end{bmatrix}$$

Diagonalization of A is given by $P^{-1}AP$.

Now, $$P^{-1}AP = \begin{bmatrix} -4 & 2 & 3 \\ -1 & 0 & 1 \\ 3 & -1 & -2 \end{bmatrix} \begin{bmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 2 & 3 \\ -1 & 0 & 1 \\ 3 & -1 & -2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 4 \\ 0 & -1 & 2 \\ 0 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = D$$

Thus $\quad \boldsymbol{P^{-1}AP = D = Diag\,(0,\ 1,\ 2)}$

Further we have $A^n = P\,D^n\,P^{-1}$

$\therefore \quad A^5 = P\,D^5\,P^{-1}$ and $D^5 = Diag\,(0^5, 1^5, 2^5) = Diag\,(0, 1, 32)$

Hence, $A^5 = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 32 \end{bmatrix} \begin{bmatrix} -4 & 2 & 3 \\ -1 & 0 & 1 \\ 3 & -1 & -2 \end{bmatrix}$

Thus $A^5 = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 96 & -32 & -64 \end{bmatrix} = \begin{bmatrix} \mathbf{191} & \mathbf{-64} & \mathbf{-127} \\ \mathbf{97} & \mathbf{-32} & \mathbf{-65} \\ \mathbf{190} & \mathbf{-64} & \mathbf{-126} \end{bmatrix}$

>> **Referring to problem-6**, we have the eigen values of A, $\lambda_1 = 0, \lambda_2 = 3, \lambda_3 = 15$

and the corresponding eigen vectors are

$$X_1 = (1, 2, 2)', \quad X_2 = (2, 1, -2)', \quad X_3 = (2, -2, 1)'$$

Hence the modal matrix $P = [X_1, X_2, X_3] = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$

$$|P| = 1(1-4) - 2(2+4) + 2(-4-2) = -27$$

$$Adj\ P = \begin{bmatrix} +(1-4), & -(2+4), & +(-4-2) \\ -(2+4), & +(1-4), & -(-2-4) \\ +(-4-2), & -(-2-4), & +(1-4) \end{bmatrix} = \begin{bmatrix} -3 & -6 & -6 \\ -6 & -3 & 6 \\ -6 & 6 & -3 \end{bmatrix}$$

$$P^{-1} = \frac{1}{|P|}(Adj\ P)$$

$$P^{-1} = \frac{1}{-27} \begin{bmatrix} -3 & -6 & -6 \\ -6 & -3 & 6 \\ -6 & 6 & -3 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

Diagonalization of A is given by $P^{-1}AP$.

Now, $P^{-1}AP = \frac{1}{9} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$

$$= \frac{1}{9} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 6 & 30 \\ 0 & 3 & -30 \\ 0 & -6 & 15 \end{bmatrix}$$

ie., $P^{-1}AP = \frac{1}{9}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 135 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix} = D$

Thus $\mathbf{P^{-1}\,AP = D = Diag\,(0,\ 3,\ 15)}$

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18. *Show that the matrix* $A = \begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{bmatrix}$ *is similar to its diagonal matrix. Find the associated diagonal matrix.*

>> *Referring to Problem-7,* we have the eigen values of A, $\lambda_1 = 3$, $\lambda_2 = 6$, $\lambda_3 = 9$ and the corresponding eigen vectors are,

$$X_1 = (1, 2, 2)',\ X_2 = (2, 1, -2)',\ X_3 = (2, -2, 1)'$$

Hence the modal matrix $P = [X_1\ X_2\ X_3] = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$

Remark : *The matrix P is same as in the previous problem and hence P^{-1} is also same as in the previous problem.*

Diagonalization of A is given by $P^{-1}AP$.

Now, $P^{-1}AP = \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & -1 \end{bmatrix}\begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$

$$= \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 12 & 18 \\ 6 & 6 & -18 \\ 6 & -12 & 9 \end{bmatrix}$$

$$P^{-1}AP = \frac{1}{9}\begin{bmatrix} 27 & 0 & 0 \\ 0 & 54 & 0 \\ 0 & 0 & 81 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \end{bmatrix} = D$$

Thus $\mathbf{P^{-1}AP = D = Diag\,(3,\ 6,\ 9)}$

>> **Referring to Problem-11,** we have the eigen values of A,

$$\lambda_1 = -2, \ \lambda_2 = 3, \ \lambda_3 = 6$$

and the corresponding eigen vectors are

$$X_1 = (1, 0, -1)', \ X_2 = (1, -1, 1)', \ X_3 = (1, 2, 1)'$$

Hence the **Modal matrix** $P = [X_1 \ X_2 \ X_3] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ -1 & 1 & 1 \end{bmatrix}$

Now, $|P| = 1(-1-2)-1(2+1) = -6$ *(Expanded by first column)*

$$Adj\ P = \begin{bmatrix} +(-1-2), & -(1-1), & +(2+1) \\ -(0+2), & +(1+1), & -(2-0) \\ +(0-1), & -(1+1), & +(-1-0) \end{bmatrix} = \begin{bmatrix} -3 & 0 & 3 \\ -2 & 2 & -2 \\ -1 & -2 & -1 \end{bmatrix}$$

$$P^{-1} = \frac{1}{|P|}(Adj\ P) = \frac{-1}{6}\begin{bmatrix} -3 & 0 & 3 \\ -2 & 2 & -2 \\ -1 & -2 & -1 \end{bmatrix}$$

Diagonalization of A is given by $P^{-1}AP$

Now, $P^{-1}AP = \frac{-1}{6}\begin{bmatrix} -3 & 0 & 3 \\ -2 & 2 & -2 \\ -1 & -2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ -1 & 1 & 1 \end{bmatrix}$

$$= \frac{-1}{6}\begin{bmatrix} -3 & 0 & 3 \\ -2 & 2 & -2 \\ -1 & -2 & -1 \end{bmatrix}\begin{bmatrix} -2 & 3 & 6 \\ 0 & -3 & 12 \\ 2 & 3 & 6 \end{bmatrix}$$

$$P^{-1}AP = \frac{-1}{6}\begin{bmatrix} 12 & 0 & 0 \\ 0 & -18 & 0 \\ 0 & 0 & -36 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6. \end{bmatrix} = D$$

Thus **Spectral matrix of** $A = D = Diag(-2, 3, 6)$

>> **Referring to problem - 9** we have the eigen values of A $\lambda = -3, -3, 5$

The eigen vector corresponding to the coincident eigen value $\lambda = -3$ be denoted by $X_{1,2}$ and we have $X_{1,2} = (3k_1 - 2k_2, k_2, k_1)'$ where k_1, k_2 are arbitrary. We choose convenient values for k_1 and k_2 to obtain two distinct eigen vectors.

(i) Let $k_1 = 1,\ k_2 = 1 \quad \therefore \quad X_1 = (1, 1, 1)'$

(ii) Let $k_1 = 1,\ k_2 = 0 \quad \therefore \quad X_2 = (3, 0, 1)'$

Further we have obtained (*Problem*-9) the eigen vector corresponding to $\lambda = 5$ as $(1, 2, -1)'$

Denoting $X_3 = (1, 2, -1)'$, we have modal matrix

$$P = [X_1 X_2 X_3] = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

$$|P| = 1(-2) - 3(-3) + 1(1) = 8$$

$$Adj\ P = \begin{bmatrix} +(0-2), & -(-3-1), & +(6-0) \\ -(-1-2), & +(-1-1), & -(2-1) \\ +(1-0), & -(1-3), & +(0-3) \end{bmatrix}$$

Diagonalization of A is given by $P^{-1}AP$

Now, $$P^{-1}AP = \frac{1}{8}\begin{bmatrix} -2 & 4 & 6 \\ 3 & -2 & -1 \\ 1 & 2 & -3 \end{bmatrix}\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 3 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} -2 & 4 & 6 \\ 3 & -2 & -1 \\ 1 & 2 & -3 \end{bmatrix}\begin{bmatrix} -3 & -9 & 5 \\ -3 & 0 & 10 \\ -3 & -3 & -5 \end{bmatrix}$$

$$P^{-1}AP = \frac{1}{8}\begin{bmatrix} -24 & 0 & 0 \\ 0 & -24 & 0 \\ 0 & 0 & 40 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix} = D$$

Thus $P^{-1}AP = D = Diag(-3, -3, 5)$

20 Determine the [illegible]

$A =$ [illegible]

>> **Referring to problem-11**, we have the eigen values of A, $\lambda = 2, 2, 8$

The eigen vector corresponding to the coincident eigen values $\lambda = 2$ be denoted by $X_{1,2}$ and we have $X_{1,2} = \left[\dfrac{k_2 - k_1}{2}, k_2, k_1\right]'$ where k_1, k_2 are arbitrary.

We choose convenient values for k_1 and k_2 to obtain two distinct eigen vectors which are orthogonal.

(i) Let $k_1 = 1, k_2 = 1 : X_1 = [0, 1, 1]'$

(ii) Suppose $X_2 = [a, b, c]'$ then we must have $X_1' \cdot X_2' = 0$

i.e., $\quad 0 + b + c = 0$ or $b = -c$ or $\dfrac{b}{1} = \dfrac{c}{-1}$.

Since a is arbitrary, let us choose $a = 1$

$\therefore \quad X_2 = (1, 1, -1)'$ and we observe $X_1' \cdot X_2' = 0$

Further we have obtained the eigen vector corresponding to $\lambda = 8$ as $(2, -1, 1)'$

Denoting $X_3 = (2, -1, 1)'$ we also observe that $X_2' \cdot X_3' = 0$ and $X_3' \cdot X_1' = 0$

The modal matrix $P = [X_1 X_2 X_3] = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

$$|P| = -1(1+1) + 2(-1-1) = -6$$

$$Adj\ P = \begin{bmatrix} +(1-1), & -(1+2), & +(-1-2) \\ -(1+1), & +(0-2), & -(0-2) \\ +(-1-1), & -(0-1), & +(0-1) \end{bmatrix} = \begin{bmatrix} 0 & -3 & -3 \\ -2 & -2 & 2 \\ -2 & 1 & -1 \end{bmatrix}$$

$$P^{-1} = \frac{1}{|P|}(Adj\,P) = \frac{-1}{6}\begin{bmatrix} 0 & -3 & -3 \\ -2 & -2 & 2 \\ -2 & 1 & -1 \end{bmatrix}$$

Diagonalization of A is given by $P^{-1}AP$

$$P^{-1}AP = \frac{-1}{6}\begin{bmatrix} 0 & -3 & -3 \\ -2 & -2 & 2 \\ -2 & 1 & -1 \end{bmatrix}\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$= \frac{-1}{6}\begin{bmatrix} 0 & -3 & -3 \\ -2 & -2 & 2 \\ -2 & 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 2 & 16 \\ 2 & 2 & -8 \\ 2 & -2 & 8 \end{bmatrix}$$

$$= \frac{-1}{6}\begin{bmatrix} -12 & 0 & 0 \\ 0 & -12 & 0 \\ 0 & 0 & -48 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix} = D$$

Thus $P^{-1}AP = D = Diag(2, 2, 8)$

Remark :

1. *If the orthogonal eigen vectors X_1, X_2, X_3 are normalized then the associated modal matrix* $P = \begin{bmatrix} 0 & 1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$ *is an orthogonal matrix which also will give us* $P^{-1}AP = P'AP = Diag(2, 2, 8)$.
2. *If orthogonal congruence was not specified we can arbitrarily choose k_1 and k_2 in $X_{1,2}$ to obtain two linearly independent eigen vectors (Similar to the previous problem) X_1 and X_2. Along with X_3 the associated modal matrix P will also give us* $P^{-1}AP = Diag(2, 2, 8)$

>> The characteristic equation of A is $|A - \lambda I| = 0$

ie.,
$$\begin{vmatrix} (2-\lambda) & 1 & 0 \\ 0 & (2-\lambda) & 1 \\ 0 & 0 & (2-\lambda) \end{vmatrix} = 0$$

ie., $(2-\lambda)^3 = 0 \Rightarrow \lambda = 2, 2, 2.$

The eigen vector corresponding to $\lambda = 2$ has to be obtained by solving the system of equations.

$$(2-2)x + 1y + 0z = 0$$
$$0x + (2-2)y + 1z = 0$$
$$0x + 0y + (2-2)z = 0$$

ie., $y = 0,\ z = 0$; x can be arbitrary.

$\therefore$ $x = k,\ y = 0,\ z = 0$ is the eigen vector corresponding to the coincident eigen value $\lambda = 2$.

It is evident that we cannot obtain three linearly independent eigen vectors.

Thus we conclude that the matrix A is not diagonizable.

>> Since the eigen vectors of a symmetric matrix are orthogonal we shall form the modal matrix with normalized eigen vectors.

$$\text{Hence } P = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

By data, the eigen values of A are $\lambda_1 = 0,\ \lambda_2 = 3,\ \lambda_3 = 15$

$\therefore$ Diagonal matrix $D = Diag\ (0, 3, 15)$

We know that $D = P^{-1} AP$

Premultiplying by P and post multiplying by P^{-1} we have,

$$PDP^{-1} = PP^{-1}\ A\ PP^{-1} = IAI = A$$

But $P^{-1} = P'$ since P is orthogonal.

$\therefore$ $A = PDP'$

$$= \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 6 & 3 & -6 \\ 30 & -30 & 15 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 72 & -54 & 18 \\ -54 & 63 & -36 \\ 18 & -36 & 27 \end{bmatrix} = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Thus the required symmetric matrix $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Remark :

1. *Since we had normalized eigen vectors in P, P was an orthogonal matrix and hence we could use $P^{-1} = P'$ in the computation of $A = P D P^{-1}$. If P was formed with the actual eigen vectors, it would have been necessary to compute P^{-1} in the process of finding the symmetric matrix A.*
2. *Compare this problem with the earlier worked problem – 17*

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22. *If* $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ *is the given matrix and* $P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ *is the modal matrix find the value of θ which reduces A to the diagonal matrix*

>> We have, $P^{-1} AP = D$

Here $|P| = 1$, $Adj\ P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = P^{-1}$ since $|P| = 1$

$$P^{-1} AP = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a\cos\theta - b\sin\theta & a\sin\theta + b\cos\theta \\ b\cos\theta - c\sin\theta & b\sin\theta + c\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\,(a\cos\theta - b\sin\theta) - \sin\theta\,(b\cos\theta - c\sin\theta) & \cos\theta\,(a\sin\theta + b\cos\theta) - \sin\theta\,(b\sin\theta + c\cos\theta) \\ \sin\theta\,(a\cos\theta - b\sin\theta) + \cos\theta\,(b\cos\theta - c\sin\theta) & \sin\theta\,(a\sin\theta + b\cos\theta) + \cos\theta\,(b\sin\theta + c\cos\theta) \end{bmatrix}$$

$$P^{-1} AP = \begin{bmatrix} (a\cos^2\theta - b\sin 2\theta + c\sin^2\theta) & (a-c)\sin\theta\cos\theta + b\cos 2\theta \\ (a-c)\sin\theta\cos\theta + b\cos 2\theta & a\sin^2\theta + b\sin 2\theta + c\cos^2\theta \end{bmatrix}$$

$$P^{-1} AP = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} = D$$

This clearly implies that we must have,

$$(a-c)\sin\theta\cos\theta + b\cos 2\theta = 0$$

or $\quad (a-c)\dfrac{\sin 2\theta}{2} = -b\cos 2\theta$

or $\quad \dfrac{\sin 2\theta}{\cos 2\theta} = \dfrac{-2b}{a-c}$

ie., $\quad \tan 2\theta = \dfrac{2b}{c-a} \Rightarrow 2\theta = \tan^{-1}\left(\dfrac{2b}{c-a}\right)$

Thus the required $\boldsymbol{\theta = \frac{1}{2}\tan^{-1}\left(\frac{2b}{c-a}\right)}$

8.5 Quadratic Forms

A homogeneous expression of second degree in any number of variables is called a ***Quadratic Form***. (Q.F)

Examples :

1. $2x^2 + 3xy + 4y^2$
2. $x_1^2 + 2x_2^2 - 3x_3^2 + 4x_1 x_2 - x_2 x_3 + 6x_3 x_1$

In general we have,

$$a_{11} x_1^2 + 2a_{12} x_1 x_2 + a_{22} x_2^2 \quad \ldots \text{(i)}$$

$$a_{11} x_1^2 + a_{22} x_2^2 + a_{33} x_3^2 + 2 a_{12} x_1 x_2 + 2 a_{23} x_2 x_3 + 2 a_{31} x_3 x_1 \quad \ldots \text{(ii)}$$

respectively representing quadratic forms in two and three variables.

It is possible to represent a quadratic form as a product of three matrices in the form $X' AX$ where X is the column matrix in the variables, A is a symmetric matrix and X' being the transpose of X is a row matrix.

With reference to (i) we have,

$$X' A X = [x_1 \, x_2] \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

With reference to (ii) we have,

$$X' A X = [x_1 \, x_2 \, x_3] \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

A is called the matrix of the quadratic form. We can easily write the matrix A of a given quadratic form and conversely given a symmetric matrix. We can write the associated quadratic form.

The symmetric matrix A associated with (i) and (ii) can be written as follows.

(i) $$A = \begin{bmatrix} \text{coeff. of } x_1^2 & \frac{1}{2} \text{coeff. of } x_1 x_2 \\ \frac{1}{2} \text{coeff of } x_1 x_2 & \text{coeff. of } x_2^2 \end{bmatrix}$$

(ii) $$A = \begin{bmatrix} \text{coeff. of } x_1^2 & \frac{1}{2}\text{coeff. of } x_1 x_2 & \frac{1}{2}\text{coeff. of } x_1 x_3 \\ \frac{1}{2}\text{coeff. of } x_1 x_2 & \text{coeff. of } x_2^2 & \frac{1}{2}\text{coeff. of } x_2 x_3 \\ \frac{1}{2}\text{coeff. of } x_1 x_3 & \frac{1}{2}\text{coeff. of } x_2 x_3 & \text{coeff. of } x_3^2 \end{bmatrix}$$

Illustrative Examples

1. $x_1^2 + 2x_2^2 + 3x_3^2 + 4x_1 x_2 + 6x_1 x_3 + 8x_2 x_3$

>> $$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \\ 3 & 4 & 3 \end{bmatrix}$$

2. $6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1 x_2 - 2x_2 x_3 + 4x_3 x_1$

>> $$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

3. $x^2 + 2y^2 + 3z^2 + 4xy + 5yz + 6zx$

>> $$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 5/2 \\ 3 & 5/2 & 3 \end{bmatrix}$$

4. $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$

>> $$A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

5. $16x_1^2 - x_2^2 + 3x_1 x_3 - 6x_2 x_3$

>> $$A = \begin{bmatrix} 16 & 0 & 3/2 \\ 0 & -1 & -3 \\ 3/2 & -3 & 0 \end{bmatrix}$$

6. $xy + yz + zx$

>> $$A = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

Further given a symmetric matrix, we can write the associated quadratic form easily.

Illustrative Examples

1. $ax^2 + 2hxy + by^2$

>> $A = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$

2. $x^2 + y^2 + z^2 + 4xy + 6yz - 2zx$

>> $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 3 \\ -1 & 3 & 1 \end{bmatrix}$

3. $x_1^2 + x_2^2 + x_3^2 + 4x_1 x_2 + 8x_1 x_3$

>> $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$

8.51 Reduction of a quadratic form into canonical form

Let $X' A X$ be the given quadratic form where A is a symmetric matrix.

Consider a linear transformation $X = PY$

Then $X' A X = (PY)' A (PY)$

$= (Y' P') A (PY)$

$= Y' (P' A P) Y$

or $X' AX = Y' BY$ where $B = P' AP$

If $B = P' AP$ then B and A are ***congruent matrices***. Further the transformation $X = PY$ is called a ***congruent transformation***.

Canonical form : Rank, Index and Signature

If $B = P' AP$ is a diagonal matrix, then the transformed quadratic form $Y' BY$ is a sum of square terms known as ***canonical form***. $X' AX$ is transformed into the form $d_1 y_1^2 + d_2 y_2^2 + \cdots + d_n y_n^2$ being the canonical form.

The *rank (r) of B or A is called the* ***rank of the quadratic form.***

The number of positive terms in the canonical form of a quadratic form is known as the ***Index (p) of the quadratic form***.

The difference between the number of positive terms and negative terms in the canonical form is known as the ***signature*** of the quadratic form.

Note : $B = Diag\ (d_1, d_2, \ldots d_n)$ *can further be reduced to* $B = Diag\ (\pm 1, \pm 1, \pm 1, \ldots \pm 1)$ *and the associated canonical form will be* $\pm y_1^2 \pm y_2^2 \pm \ldots \pm y_n^2$.

8.52 Nature of quadratic form

If r *is the rank and* p *is the index of the quadratic form in* n variables the nature of the quadratic form is identified as presented in the following table.

	Condition	Nature of Q.F	Canonical form	Remark on canonical form
1.	$r = n, p = n$	Positive definite	$y_1^2 + y_2^2 + \cdots + y_n^2$	Only positive terms (n terms).
2.	$r = n, p = 0$	Negative definite	$-y_1^2 - y_2^2 \ldots - y_n^2$	Only negative terms (n terms)
3.	$r = p, p < n$	Positive semi definite	$y_1^2 + y_2^2 + \cdots + y_r^2$	Only positive terms (r terms)
4.	$r < n, p = 0$	Negative semi-definte	$-y_1^2 - y_2^2 - \ldots - y_r^2$	Only negative terms (r terms)

In all other cases the quadratic form is said to be **indefinite**. Indefinite quadratic form will contain both positive and negative terms in the canonical form.

Note : ***Orthogonal Transformation***

Suppose $\lambda_1, \lambda_2, \lambda_3$ *are the eigen values of A having corresponding orthogonal eigen vectors* X_1, X_2, X_3 *in the normalized form, the associated modal matrix P will be an orthogonal matrix.* ($P^{-1} = P'$ *is this case)*

We have in this case,

$$P^{-1}AP = P'AP = D = Diag\,(\lambda_1, \lambda_2, \lambda_3)$$

The associatead canonical form will be $\lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2$

Accordingly the nature of the quadratic form is presented in the following table.

	Nature of quadratic form	Nature of eigen values
1.	Positive definite	Positive eigen values
2.	Negative definite	Negative eigen values
3.	Positive semi definite	Positive eigen values at least one is zero.
4.	Negative semi definite	Negative eigen values atleast one is zero.

In the case of indefinite quadratic form there will be positive as well as negative eigen values.

Working procedure for problems to reduce the given quadratic form to sum of squares (canonical form)

Case-(i) ***By canonical transformation***

- We write the matrix A of the Q.F.
- We write $A = I\,A\,I$
- We perform elementary transformation to reduce A to the diagonal form
- The elementary row transformations are also performed on the premultiplied I where as the column transformations are performed on the post multiplied I.
- We obtain $D = P'AP$
- If $D = Diag\,(d_1, d_2, d_3)$ in respect of a third order square matrix A, the canonical form of the given Q.F is $d_1 y_1^2 + d_2 y_2^2 + d_3 y_3^2$
- $X = PY$ where $Y = [\,y_1\, y_2\, y_3\,]$ and $X = [\,x_1\, x_2\, x_3\,]'$ will give us the congruent linear transformation.
- If required we can also obtain the canonical form as $\pm y_1^2 \pm y_2^2 \pm y_3^2$

Case-(ii) ***By orthogonal transformation***

- We write the matrix A of the Q.F
- We obtain the eigen values $\lambda_1, \lambda_2, \lambda_3$ and the corresponding orthogonal eigen vectors X_1, X_2, X_3 of the third order square matrix A.
- We normalize the orthogonal vectors X_1, X_2, X_3 and write the associated orthogonal modal matrix P.
- Since $P^{-1} = P'$ in this case we have $P'AP = Diag\,(\lambda_1, \lambda_2, \lambda_3)$
- The associated canonical form is $\lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2$
- $X = PY$ where $Y = [\,y_1, y_2, y_3\,]'$ and $X = [\,x_1, x_2, x_3\,]'$ will give us the orthogonal linear transformation.

WORKED PROBLEMS

24 *Reduce the following quadratic form to canonical form by (a) Congruent transformation (b) Orthogonal transformation*

$$2x_1^2 + 2x_2^2 + 2x_3^2 + 2x_1 x_3$$

>> The symmetric matrix A of the given Q.F is $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

(a) *By congruent transformation*

Let $A = I A I$

ie.,
$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to -1/2 \cdot R_1 + R_3$

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to -1/2 \cdot C_1 + C_3$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

ie., $D = P' AP$

where $P = \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

The canonical form is $Y' DY$ where $Y = [y_1 \; y_2 \; y_3]'$

ie.,
$$Y' DY = [y_1 \; y_2 \; y_3] \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3/2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Thus $2y_1^2 + 2y_2^2 + (3/2)\, y_3^2$ *is the canonical form* of the given quadratic form.

The associatead congruent linear transformation is $X = PY$

ie., $x_1 = y_1 - (1/2)\, y_2, \; x_2 = y_2, \; x_3 = y_3$

(b) *By orthogonal transformation*

We have to first compute the eigen values and the corresponding eigen vector of A.

Referring to problem-8, we have obtained

$$\lambda_1 = 1, \; \lambda_2 = 2, \; \lambda_3 = 3$$

$$X_1 = (-1, 0, 1)', \; X_2 = (0, 1, 0)', \; X_3 = (1, 0, 1)'$$

We normalize these vectors and write the associated modal matrix P.

ie., $P = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$

The orthogonal transformation $X = PY$ transforms the quadrataic form $X'AX$ into $Y'DY$ where $D = P^{-1}AP = P'AP$ is the diagonal matrix given by

$$D = Diag(\lambda_1, \lambda_2, \lambda_3) = Diag(1, 2, 3)$$

Thus $y_1^2 + 2y_2^2 + 3y_3^2$ is the canonical form of the given Q.F.

The associated orthogonal linear transformation $X = PY$ is given by

$$x_1 = (-1/\sqrt{2})\,y_1 + (1/\sqrt{2})\,y_3,\quad x_2 = y_2,\quad x_3 = (1/\sqrt{2})\,y_1 + (1/\sqrt{2})\,y_3$$

Remark :

1. *Rank of the Q.F = Rank of A = 3*

   Index of the Q.F = No. of positive terms in the canonicl form = 3

   Signature of the Q.F = Difference between the number of positive and negative terms $= 3 - 0 = 3$

   Nature of the Q.F is positive definite.

2. *If orthogonal transformation is not specified, we always adopt congrue transformation to reduce the given Q.F into sum of squares.*

---

25. *Find the transformation which will transform the following quadratic form into sum of squares and find the reduced form*

$$4x^2 + 3y^2 + z^2 - 8xy - 6yz + 4xz$$

>> The symmetric matrix A of the given Q.F is $A = \begin{bmatrix} 4 & -4 & 2 \\ -4 & 3 & -3 \\ 2 & -3 & 1 \end{bmatrix}$

Let $A = IAI$

ie., $\begin{bmatrix} 4 & -4 & 2 \\ -4 & 3 & -3 \\ 2 & --3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$R_2 \to R_1 + R_2,\; R_3 \to -1/2 \cdot R_1 + R_3$

$$\begin{bmatrix} 4 & -4 & 2 \\ 0 & -1 & -1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to C_1 + C_2,\; C_3 \to -1/2 \cdot C_1 + C_3$

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ -1/2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to -R_2 + R_3$

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -3/2 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to -C_2 + C_3$

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -3/2 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -3/2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

We have $D = Diag\,(4, -1, 1) = P'AP$, where $P = \begin{bmatrix} 1 & 1 & -3/2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

$X = PY$ is the congruent transformation which has transformed the given quadratic form into sum of squares given by,

$$4u^2 - v^2 + w^2$$

Where $Y = [\,u\,v\,w\,]'$ and $X = [\,x\,y\,z\,]'$

The **congruent transformation** is given by $x = u + v - (3/2)\,w,\ y = v - w,\ z = w$

Remark : ***Nature of the quadratic form***

Rank = 3; *Index* = 2, *Signature* = 2 − 1 = 1

Q.F is indefinite since the canonical form has both positive and negative terms.

>> The symmetric matrix A of the given Q.F is

$$A = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 2 & 4 \\ 0 & 4 & -7 \end{bmatrix}$$

Let $A = I\,A\,I$

ie.,

$$\begin{bmatrix} 1 & -2 & 0 \\ -2 & 2 & 4 \\ 0 & 4 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to 2R_1 + R_2$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & -2 & 4 \\ 0 & 4 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \rightarrow 2C_1 + C_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 4 \\ 0 & 4 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \rightarrow 2R_2 + R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 \rightarrow 2C_2 + C_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

We have $D = Diag\,(1, -2, 1) = P'AP$, where $P = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

$X = PY$ is the congruent transformation where $X = [x_1\ x_2\ x_3]'$ and $Y = [y_1\ y_2\ y_3]'$

ie., $x_1 = y_1 + 2y_2 + 4y_3$, $x_2 = y_2 + 2y_3$, $x_3 = y_3$

Canonical form is $y_1^2 - 2y_2^2 + y_3^2$

Remark : ***Nature of the quadratic form***

Rank = 3, *Index* = 2, *Signature* = 1, *Indefinite form.*

>> The symmetric matrix A of the given form is $A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -2 & -2 \\ 3 & -2 & 3 \end{bmatrix}$

Let $A = I A I$

ie., $$\begin{bmatrix} 1 & 0 & 3 \\ 0 & -2 & -2 \\ 3 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to -3R_1 + R_3$

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & -2 & -2 \\ 0 & -2 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to -3C_1 + C_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & -2 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to -R_2 + R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to -C_2 + C_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Further we need to make the diagonal elements 1 numerically. Hence we perform the transformations.

$$\frac{1}{\sqrt{2}} R_2, \; \frac{1}{\sqrt{2}} C_2 \text{ and } \frac{1}{2} R_3, \; \frac{1}{2} C_3$$

ie., $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 \\ -3/2 & -1/2 & 1/2 \end{bmatrix} A \begin{bmatrix} 1 & 0 & -3/2 \\ 0 & 1/\sqrt{2} & -1/2 \\ 0 & 0 & 1/2 \end{bmatrix}$$

We have $D = Diag(1, -1, -1) = P' AP$

The canonical form is $y_1^2 - y_2^2 - y_3^2$ under the congruent transformation $X = PY$,

where $X = [x \; y \; z]'$, $Y = [y_1 \; y_2 \; y_3]'$ and $P = \begin{bmatrix} 1 & 0 & -3/2 \\ 0 & 1/\sqrt{2} & -1/2 \\ 0 & 0 & 1/2 \end{bmatrix}$

The congruent transformation is $x = y_1 - (3/2)\,y_2$, $y = (1/\sqrt{2})\,y_2 - (1/2)\,y_3$, $z = (1/2)\,y_3$

The **quadratic form is indefinite** as the canonical form contains both positive and negative terms.

28. [illegible]

\>> The symmetric matrix A of the given Q.F is

$$A = \begin{bmatrix} 2 & -1 & 3 \\ -1 & -2 & -4 \\ 3 & -4 & 2 \end{bmatrix}$$

Let $A = I\,A\,I$

ie.,
$$\begin{bmatrix} 2 & -1 & 3 \\ -1 & -2 & -4 \\ 3 & -4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to 1/2 \cdot R_1 + R_2$, $R_3 \to -3/2 \cdot R_1 + R_3$

$$\begin{bmatrix} 2 & -1 & 3 \\ 0 & -5/2 & -5/2 \\ 0 & -5/2 & -5/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ -3/2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to 1/2 \cdot C_1 + C_2$; $C_3 \to -3/2 \cdot C_1 + C_3$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -5/2 & -5/2 \\ 0 & -5/2 & -5/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ -3/2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1/2 & -3/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to -R_2 + R_3$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -5/2 & -5/2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ -2 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1/2 & -3/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to -C_2 + C_3$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -5/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ -2 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1/2 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

We have $D = Diag\,(2\,,\,-5/2\,,\,0) = P'\,AP$

The canonical form is $2y_1^2 - (5/2)\,y_2^2$ under the congruent transformation $X = PY$

where $X = [x\ y\ z]'$, $Y = [y_1\ y_2\ y_3]'$ and $P = \begin{bmatrix} 1 & 1/2 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

The congruent transformation is

$$x = y_1 + (1/2)\,y_2 - 2y_3, \quad y = y_2 - y_3, \quad z = y_3$$

The quadratic form has,

Rank = 2, Index = 1, Signature = 0 and the nature is indefinite.

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

>> Let $A = I\,A\,I$

ie.,
$$\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to 1/3 \cdot R_1 + R_2,\ R_3 \to -1/3 \cdot R_1 + R_3$

$$\begin{bmatrix} 6 & -2 & 2 \\ 0 & 7/3 & -1/3 \\ 0 & -1/3 & 7/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ -1/3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to 1/3 \cdot C_1 + C_2,\ C_3 \to -1/3 \cdot C_1 + C_3$

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 7/3 & -1/3 \\ 0 & -1/3 & 7/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ -1/3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1/3 & -1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to 1/7 \cdot R_2 + R_3$

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 7/3 & -1/3 \\ 0 & 0 & 16/7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ -2/7 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1/3 & -1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to 1/7 \cdot C_2 + C_3$

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 7/3 & 0 \\ 0 & 0 & 16/7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ -2/7 & 1/7 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1/3 & -2/7 \\ 0 & 1 & 1/7 \\ 0 & 0 & 1 \end{bmatrix}$$

We have $D = Diag(6, 7/3, 16/7) = P'AP$, where $P \begin{bmatrix} 1 & 1/3 & -2/7 \\ 0 & 1 & 1/7 \\ 0 & 0 & 1 \end{bmatrix}$

The quadratic form $6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 + 4x_1x_3 - 2x_2x_3$ is reduced to the canonical form

$$6y_1^2 + (7/3)y_2^2 + (16/7)y_3^2$$

under the transformation $X = PY$ given by :

$$x_1 = y_1 + (1/3)y_2 - (2/7)y_3, \quad x_2 = y_2 + (1/7)y_3, \quad x_3 = y_3$$

Further the quadratic form has,

Rank = 3, Index = 3, Signature = 3 and is positive definite in nature.

Remark : *Canonical form of this quadrataic form by orthogonal transformation.*

Referring to problem – 20 and the Remark made in that problem we have

$P'AP = Diag(2, 2, 8)$ where P is the orthogonal matrix given by

$$P = \begin{bmatrix} 0 & 1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$$

The canonical form by orthogonal transformation $X = PY$ is $2y_1^2 + 2y_2^2 + 8y_3^2$.
The rank, index, signature and the nature is the same as stated earlier.

>> The symmetric matrix A of the given Q.F is

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

Let $A = IAI$

ie., $\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$R_2 \to -1/3 \cdot R_1 + R_2, \; R_3 \to -1/3 \cdot R_1 + R_3$

$$\begin{bmatrix} 3 & 1 & 1 \\ 0 & 8/3 & -4/3 \\ 0 & -4/3 & 8/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1/3 & 1 & 0 \\ -1/3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to -1/3 \cdot C_1 + C_2, \; C_3 \to -1/3 \cdot C_1 + C_3$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 8/3 & -4/3 \\ 0 & -4/3 & 8/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1/3 & 1 & 0 \\ -1/3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1/3 & -1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to 1/2 \cdot R_2 + R_3$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 8/3 & -4/3 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1/3 & 1 & 0 \\ -1/2 & 1/2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1/3 & -1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to 1/2 \cdot C_2 + C_3$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 8/3 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1/3 & 1 & 0 \\ -1/2 & 1/2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1/3 & -1/3 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$$

We have $D = Diag(3, 8/3, 2) = P'AP$ where $P \begin{bmatrix} 1 & -1/3 & -1/3 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$

The canonical form is $3y_1^2 + (8/3)y_2^2 + 2y_3^2$ under the congruent transformation.

$X = PY$ where $X = [x \; y \; z]'$ and $y = [y_1 \; y_2 \; y_3]'$.

The quadratic form is positive definite.

>> The symmetric matrix A of the Q.F is

$$A = \begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix}$$

Let $A = I\ A\ I$

ie., $\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$R_2 \to -3/5 \cdot R_1 + R_2,\ R_3 \to -7/5 \cdot R_1 + R_3$

$$\begin{bmatrix} 5 & 3 & 7 \\ 0 & 121/5 & -11/5 \\ 0 & -11/5 & 1/5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3/5 & 1 & 0 \\ -7/5 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to -3/5 \cdot C_1 + C_2,\ C_3 \to -7/5 \cdot C_1 + C_3$

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 121/5 & -11/5 \\ 0 & -11/5 & 1/5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3/5 & 1 & 0 \\ -7/5 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -3/5 & -7/5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to 1/11 \cdot R_2 + R_3$

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 121/5 & -11/5 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3/5 & 1 & 0 \\ -16/11 & 1/11 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -3/5 & -7/5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to 1/11\ C_2 + C_3$

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 121/5 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3/5 & 1 & 0 \\ -16/11 & 1/11 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -3/5 & -16/11 \\ 0 & 1 & 1/11 \\ 0 & 0 & 1 \end{bmatrix}$$

We have $D = Diag\ (5,\ 121/5,\ 0) = P' A P$ where

$$P = \begin{bmatrix} 1 & -3/5 & -16/11 \\ 0 & 1 & 1/11 \\ 0 & 0 & 1 \end{bmatrix}$$

The canonical form is $5\,y_1^2 + (121/5)\ y_2^2$

The quadratic form has,

Rank = 2, Index = 2

Since the Rank = Index = 2 < 3 the quadratic form is positve semidefinite.

The congruent transformation $X = PY$ is given by

$$x_1 = y_1 - (3/5)\ y_2 - (16/11)\ y_3,\ \ x_2 = y_2 + (1/11)\ y_3,\ \ x_3 = y_3$$

Further $y_1 = 0$ and $y_2 = 0$ will reduce the quadratic form to zero. y_3 can be arbitrary, $y_3 = 1$ (say)

Hence $x_1 = -16/11$, $x_2 = 1/11$, $x_3 = 1$ is a set of non zero values that makes the quadratic form zero.

---

32. Reduce the following quadratic form into canonical form by orthogonal transformation. Also find the rank, index, signature and the nature of the quadratic form

$$8x^2 + 7y^2 + 3z^2 - 12xy + 4xz - 8yz$$

Indicate the orthogonal transformation also

>> The symmetric matrix of the Q.F is

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Referring to problem - 17 we have,

$\lambda_1 = 0$, $\lambda_2 = 3$, $\lambda_3 = 15$ and the corresponding eigen vectors are

$X_1 = [1, 2, 2]'$, $X_2 = [2, 1, -2]'$, $X_3 = [2, -2, 1]'$

The modal matrix P consisting normalized eigen vectors is

$$P = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$P^{-1} = P'$ since P is an orthogonal matrix.

We have $D = Diag\ (0, 3, 15) = P' A P$

The canonical form is $3y_2^2 + 15y_3^2$

The quadratic form has,

Rank = 2, Index = 2, Signature = 2 and it is positive semidefinite.

Further the orthogonal transformation $X = PY$ is given by

$$x = \frac{1}{3}(y_1 + 2y_2 + y_3),\ y = \frac{1}{3}(2y_1 + y_2 - 2y_3),\ z = \frac{1}{3}(2y_1 - 2y_2 + y_3)$$

---

33. *Obtain the orthogonal transformation that transforms the quadratic form*

$$x_1^2 + 5x_2^2 + x_3^2 + 2x_1x_2 + 6x_1x_3 + 2x_2x_3 \text{ into the form } \sum d_i x_i^2$$

>> The symmetric matrix A of the Q.F is

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

Referring to problem - 19 we have

$\lambda_1 = -2, \lambda_2 = 3, \lambda_3 = 6$ and the corresponding eigen vectors

$X_1 = [1, 0, -1]', X_2 = [1, -1, 1]', X_3 = [1, 2, 1]'$

The modal matrix P consisting normalized eigen vectors is

$$P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & -1/\sqrt{3} & 2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$$

$P^{-1} = P'$ since P is an orthogonal matrix.

We have $D = Diag(-2, 3, 6) = P'AP$

The canonical form is $-2y_1^2 + 3y_2^2 + 6y_3^2$

The orthogonal transformation $X = PY$ is given by

$$x_1 = \frac{1}{\sqrt{2}}y_1 + \frac{1}{\sqrt{3}}y_2 + \frac{1}{\sqrt{6}}y_3,\ x_2 = \frac{-1}{\sqrt{3}}y_2 + \frac{2}{\sqrt{6}}y_3,\ x_3 = \frac{1}{\sqrt{2}}y_1 + \frac{1}{\sqrt{3}}y_2 + \frac{1}{\sqrt{6}}y_3$$

---

34. *Write the symmetric matrix associated with the following quadratic form.*

$$x^2 + 3y^2 + 8z^2 + 4w^2 + 4xy + 6xz - 4xw + 12yz - 8yw - 12zw$$

>> $X = [x\ y\ z\ w]'$, $X'AX$ is the qudratic form in four variables where the symmetric matrix A is given by

$$A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & 3 & 6 & -4 \\ 3 & 6 & 8 & -6 \\ -2 & -4 & -6 & 4 \end{bmatrix}$$

---

55. Write down the quadratic form corresponding to the following symmetric matrix

$$A = \begin{bmatrix} 2 & -1 & 3/2 & -2 \\ -1 & -3 & -5/2 & 3 \\ 3/2 & -5/2 & 4 & 1/2 \\ -2 & 3 & 1/2 & 1 \end{bmatrix}$$

>> Let $X = [x_1 \; x_2 \; x_3 \; x_4]'$. The Q.F $X' A X$ is given by

$$2x_1^2 - 3x_2^2 + 4x_3^2 + x_4^2 - 2x_1 x_2 + 3x_1 x_3 - 4x_1 x_4 - 5x_2 x_3 + 6x_2 x_4 + x_3 x_4$$

EXERCISES

1. Show that the transformation $y_1 = 2x_1 + x_2 + x_3$, $y_2 = x_1 + x_2 + 2x_3$, $y_3 = x_1 - 2x_3$ is regular. Find the inverse transformation.

Find all the eigen values and the corresponding eigen vectors for the following matrices.

2. $\begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix}$ 3. $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ 4. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix}$

5. Find a matrix P which transforms the following matrix A to diagonal form.

   $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ Hence find A^4

6. Diagonalize the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix}$

7. Show that the matrix $A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ is similar to its diagonal matrix. Also find transforming matrix and diagonal matrix.

8. Reduce the following quadratic form into canonical form by congruent transformation and give the corresponding linear transformation.

   $$10x_1^2 + x_2^2 + x_3^2 - 6x_1 x_2 - 2x_2 x_3 + x_3 x_1$$

9. Reduce the following quadratic form into sum of squares by an orthogonal transformation. Give the matrix and nature of the form.

   $$3x_1^2 + 3x_2^2 + 3x_3^2 + 2x_1 x_2 + 2x_1 x_3 - 2x_2 x_3$$

10. Reduce to sum of squares the quadratic form :

$$x^2 + 2y^2 - 7z^2 - 4xy + 8yz$$

Find the rank, index, signature and the nature of the form.

ANSWERS

1. $x_1 = 2y_1 - 2y_2 - y_3$, $x_2 = -4y_1 + 5y_2 + 3y_3$, $x_3 = y_1 - y_2 - y_3$
2. $\lambda = 0, -2, 1$; $(10, 3, -11)$, $(4, 3, -7)$, $(1, 0, 1)$
3. $\lambda = 1, 1, 5$; $[-(2k_1 + k_2), k_1, k_2]$, $(1, 1, 1)$
4. $\lambda = 1, 1, 1$; $(k_1, 3k_1, k_2)$
5. $P = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$; $P^{-1} AP = Diag(1, 2, 3)$

   $A^4 = \begin{bmatrix} -49 & -50 & -40 \\ 65 & 66 & 40 \\ 130 & 130 & 81 \end{bmatrix}$
6. $P^{-1} AP = D = Diag(1, 2, 3)$ where, $P = \begin{bmatrix} -1 & 1 & 1 \\ -2 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$
7. $P^{-1} AP = D = Diag(1, 2, 3)$ where, $P = \begin{bmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$
8. $10y_1^2 + \frac{1}{10} y_2$; $x_1 = y_1 + \frac{3}{10} y_2$, $x_2 = y_2 + y_3$, $x_3 = y_3$
9. $y_1^2 + 4y_2^2 + 4y_3^2$; $\begin{bmatrix} -1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$, positive definite.
10. $y_1^2 - 2y_2^2 + 9y_3^2$. Rank = 3, Index = 2, Signature = 1, Indefinte form.

BEATING THE MEMORY

[Formulae, Properties and Results to be remembered from all the units at a glance]

Unit - I DIFFERENTIAL CALCULUS-1

Table of n^{th} derivatives of standard functions

	$y = f(x)$	$y_n = D^n y$
F_1	e^{ax}	$a^n e^{ax}$
F_2	a^{mx}	$(m \log a)^n a^{mx}$
F_3	$(ax+b)^m, m > n$	$m(m-1)(m-2)\ldots[m-(n-1)]a^n(ax+b)^{m-n}$
F_4	$\frac{1}{ax+b}$	$\frac{(-1)^n n! a^n}{(ax+b)^{n+1}}$
F_5	$\log(ax+b)$	$\frac{(-1)^{n-1}(n-1)! a^n}{(ax+b)^n}$
F_6	$\sin(ax+b)$	$a^n \sin\left(\frac{n\pi}{2}+ax+b\right)$
F_7	$\cos(ax+b)$	$a^n \cos\left(\frac{n\pi}{2}+ax+b\right)$
F_8	$e^{ax}\sin(bx+c)$	$(\sqrt{a^2+b^2})^n e^{ax}\sin[n\tan^{-1}(b/a)+bx+c]$
F_9	$e^{ax}\cos(bx+c)$	$(\sqrt{a^2+b^2})^n e^{ax}\cos[n\tan^{-1}(b/a)+bx+c]$

Remark :

Observe similarities in the pair of formulae F_4 & F_5 ; F_6 & F_7 ; F_8 & F_9 as it would help to remember the formulae easily.

Leibnitz theorem for the n^{th} derivative of a product

$$D^n(uv) \text{ or } (uv)_n = uv_n + nu_1 v_{n-1} + \frac{n(n-1)}{1.2} u_2 v_{n-2} + \cdots + u_n v$$

Rolle's theorem

If $f(x)$ is continuous in $[a, b]$, differentiable in (a, b) and $f(a) = f(b)$, then there exists atleast one point c in (a, b) such that $f'(c) = 0$

Lagrange's mean value theorem

If $f(x)$ is continuous in $[a, b]$, differentiable in (a, b) then there exists atleast one point c in (a, b) such that

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

Cauchy's mean value theorem

If $f(x)$ and $g(x)$ are two continuous functions in $[a, b]$, differentiable in (a, b) with $g'(x) \neq 0$ for all x in (a, b) then there exists atleast one point c in (a, b) such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

Expansion of a function $y(x)$

- *Taylor's expansion : (about $x = a$)*

$$y(x) = y(a) + (x-a)y_1(a) + \frac{(x-a)^2}{2!} y_2(a) + \frac{(x-a)^3}{3!} y_3(a) + \cdots$$

- *Maclaurin's expansion* (about $x = 0$)

$$y(x) = y(0) + x y_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \cdots$$

Unit - II DIFFERENTIAL CALCULUS - 2

Indeterminate forms

- *L Hospital's rule (for 0/0 and ∞/∞ forms)*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = \lim_{x \to a} \frac{f''(x)}{g''(x)} \text{ etc.}$$

Polar curves

- *Angle (ϕ) between the radius vector and the tangent*

$$\tan\phi = r\frac{d\theta}{dr} \quad \text{or} \quad \cot\phi = \frac{1}{r}\frac{dr}{d\theta}$$

- *Length of the perpendicular (p) from the pole to the tangent*

$$p = r\sin\phi \quad \text{or} \quad \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

- *Angle of intersection of two polar curves is given by $|\phi_1 - \phi_2|$*

If $|\phi_1 - \phi_2| = \pi/2$ or $\tan\phi_1 \cdot \tan\phi_2 = -1$

then the curves intersect each other orthogonally or at right angles.

➢ *Pedal equation (p-r equation) of a polar curve*

If θ is eliminated from the given equation $r = f(\theta)$ and $p = r\sin\phi$, where ϕ is usually a function of θ, the resulting equation in p and r is the pedal equation of the polar curve.

Radius of curvature

➢ *Curvature* : $K = \dfrac{d\psi}{ds}$, *Radius of curvature* $\rho = \dfrac{ds}{d\psi}$

➢ *Cartesian curve* :

$$[y = y(x)],\ \rho = \frac{(1+y_1^2)^{3/2}}{y_2};\ [x = x(y)],\ \rho = \frac{(1+x_1^2)^{3/2}}{x_2}$$

➢ *Parametric curve* : $[x = x(t),\ y = y(t)]$, $\rho = \dfrac{\{(\dot{x})^2+(\dot{y})^2\}^{3/2}}{\dot{x}\,\ddot{y}-\dot{y}\,\ddot{x}}$

➢ *Polar curve* : $[r = f(\theta)]$, $\rho = \dfrac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-r\,r_2}$

➢ *Pedal curve* : $[r = f(p)]$, $\rho = r\dfrac{dr}{dp}$

UNIT - III DIFFERENTIAL CALCULUS - 3

Partial Differentiation

Partial derivatives of $u(x, y)$

First order partial derivatives

$$u_x = \frac{\partial u}{\partial x} \qquad u_y = \frac{\partial u}{\partial y}$$

Second order partial derivatives

$$u_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2} \qquad u_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial y^2}$$

$$u_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y\,\partial x} \longleftarrow \text{Mixed partial derivatives} \longrightarrow u_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x\,\partial y}$$

⟶ and so-on ⟵

Further, $\dfrac{\partial^2 u}{\partial y \partial x} = \dfrac{\partial^2 u}{\partial x \partial y}$ or $u_{yx} = u_{xy}$

Differentiation of composite functions

If $u = u(x, y)$ where $x = x(t)$ and $y = y(t)$ then

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} \quad \textit{(Total derivative)}$$

If $z = z(x, y)$ where $x = x(u, v)$ and $y = y(u, v)$ then

$$\left.\begin{aligned} \frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \end{aligned}\right\} \quad \textit{(Chain rule)}$$

Jacobians

If u, v, w are all functions of x, y, z then the *jacobian* (J) is given by

$$J = \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Taylor's series expansion of $f(x, y)$ about (a, b) and about $(0, 0)$

$$f(x, y) = f(a, b) + \frac{1}{1!}\left\{(x-a)f_x(a, b) + (y-b)f_y(a, b)\right\}$$

$$+ \frac{1}{2!}\left\{(x-a)^2 f_{xx}(a, b) + 2(x-a)(y-b)f_{xy}(a, b) + (y-b)^2 f_{yy}(a, b)\right\} + \cdots$$

In particular if $(a, b) = (0, 0)$, the series is called as **Taylor's series about the origin or Maclaurin's series** given by

$$f(x, y) = f(0, 0) + \frac{1}{1!}\left\{x f_x(0, 0) + y f_y(0, 0)\right\}$$

$$+ \frac{1}{2!}\left\{x^2 f_{xx}(0, 0) + 2xy f_{xy}(0, 0) + y^2 f_{yy}(0, 0)\right\} + \ldots$$

Maxima and Minima of $f(x, y)$

Working procedure for finding extreme values of $f(x, y)$

(i) We have to first find the stationary points (x, y) such that $f_x = 0$ and $f_y = 0$

(ii) We then find the second order partial derivatives :
$A = f_{xx}, \ B = f_{xy}, \ C = f_{yy}$

We evaluate these at all the stationary points and also compute the corresponding value of $AC - B^2$

(iii) (a) A stationary point (x_0, y_0) is a maximum point if $AC - B^2 > 0$ & $A < 0$; $f(x_0, y_0)$ is a maximum value.

(b) A stationary point (x_1, y_1) is a minimum point if $AC - B^2 > 0$ & $A > 0$; $f(x_1, y_1)$ is a minimum value.

Note : *We can overlook the cases of* $AC - B^2 < 0$, $AC - B^2 = 0$, $A = 0$

Vector Differentiation

Vector differential operator 'Nabla' (∇)

$$\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k$$

If $\phi(x, y, z)$ is a scalar point function and $\vec{A}(x, y, z)$ is a vector point function, then

$\nabla\phi$	$=$ grad ϕ	$=$ Gradient of ϕ
$\nabla \cdot \vec{A}$	$=$ div $\vec{A}$	$=$ Divergence of $\vec{A}$
$\nabla \times \vec{A}$	$=$ curl $\vec{A}$	$=$ Curl of $\vec{A}$
$\nabla \cdot \nabla \phi$	$=$ div (grad ϕ)	$=$ Laplacian of $\phi = \nabla^2 \phi$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called the Laplacian operator.

Geometrical meaning of $\nabla\phi$

If $\phi(x, y, z) = c$ be the equation of a surface, then $\nabla\phi$ is a *vector normal* to the surface.

➤ $\nabla \phi \cdot \hat{n}$ is the *directional derivative* of ϕ along a given direction $\vec{A}$ where

$\vec{A} / |\vec{A}| = \hat{n}$

- The *angle between two surfaces* is equal to the angle between their normals and if this angle is equal to 90^{o} then the surfaces are said to be orthogonal to each other.
- A vector $\vec{A}$ is said to be **solenoidal** if div $\vec{A} = 0$ and **irrotational** (*conservative*) if curl $\vec{A} = \vec{0}$.
- If $\vec{A}$ is irrotational there always exists a scalar function ϕ such that $\nabla\phi = \vec{A}$ and ϕ is called the *scalar potential* of $\vec{A}$

List of vector identities

1. curl (grad ϕ) $= \vec{0}$ 2. div (curl $\vec{A}$) $= 0$
3. curl (curl $\vec{A}$) = grad (div $\vec{A}$) $- \nabla^2 \vec{A}$
4. $\nabla \cdot (\phi \vec{A}) = \phi(\nabla \cdot \vec{A}) + \nabla\phi \cdot \vec{A}$
5. $\nabla \times (\phi \vec{A}) = \phi(\nabla \times \vec{A}) + \nabla\phi \times \vec{A}$
6. div $(\vec{A} \times \vec{B}) = \vec{B} \cdot$ curl $\vec{A} - \vec{A} \cdot$ curl $\vec{B}$

Orthogonal Curvilinear Coordinates (O.C.C)

Curvilinear coordinates : (u_1, u_2, u_3) and $\vec{r} = \vec{r}(u_1, u_2, u_3)$

Scale factors and unit vectors

$$h_1 = \left|\frac{\partial \vec{r}}{\partial u_1}\right|, \; h_2 = \left|\frac{\partial \vec{r}}{\partial u_2}\right|, \; h_3 = \left|\frac{\partial \vec{r}}{\partial u_3}\right| \;;$$

$$\hat{e}_1 = \frac{1}{h_1}\frac{\partial \vec{r}}{\partial u_1}, \; \hat{e}_2 = \frac{1}{h_2}\frac{\partial \vec{r}}{\partial u_2}, \; \hat{e}_3 = \frac{1}{h_3}\frac{\partial \vec{r}}{\partial u_3}$$

Orthogonal system	Coordinates (u_1, u_2, u_3)	Transformation	Scale factors and unit vectors
Cylindrical system	(ρ, ϕ, z) [cylindrical polar coordinates]	$x = \rho\cos\phi$ $y = \rho\sin\phi$ $z = z$	$h_1 = 1 ; \hat{e}_\rho$ $h_2 = \rho ; \hat{e}_\phi$ $h_3 = 1 ; \hat{e}_z$
Spherical system	(r, θ, ϕ) [Spherical polar coordinates]	$x = r\sin\theta\cos\phi$ $y = r\sin\theta\sin\phi$ $z = r\cos\theta$	$h_1 = 1 ; \hat{e}_r$ $h_2 = r ; \hat{e}_\theta$ $h_3 = r\sin\theta ; \hat{e}_\phi$
Cartesian system	(x, y, z) [Cartesian coordinates]	$x = x$ $y = y$ $z = z$	$h_1 = 1 ; \hat{i}$ $h_2 = 1 ; \hat{j}$ $h_3 = 1 ; \hat{k}$

Expression for the Arc length and the Volume element in O.C.C

$$ds^2 = h_1^2\, du_1^2 + h_2^2\, du_2^2 + h_3^2\, du_3^2 \quad \text{[Arc length]}$$

$$dV = h_1 h_2 h_3\, du_1\, du_2\, du_3 \quad \text{[Volume element]}$$

Expression for Gradient, Divergence, Curl and Laplacian in O.C.C

Let $\psi = \psi_1(u_1, u_2, u_3)$ be a scalar point function and

$\vec{A} = \vec{A}(u_1, u_2, u_3) = A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3$ be a vector point function.

$$\text{Grad}\,\psi = \nabla\psi = \sum \frac{1}{h_1}\frac{\partial\psi}{\partial u_1}\hat{e}_1$$

$$\text{Div}\,\vec{A} = \nabla\cdot\vec{A} = \frac{1}{h_1 h_2 h_3}\sum \frac{\partial}{\partial u_1}(A_1 h_2 h_3)$$

$$\text{Curl}\,\vec{A} = \nabla\times\vec{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ A_1 h_1 & A_2 h_2 & A_3 h_3 \end{vmatrix}$$

$$\text{Laplacian of } \psi = \nabla^2\psi = \frac{1}{h_1 h_2 h_3}\sum \frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial\psi}{\partial u_1}\right)$$

Differentiation under the Integral sign

➤ **Leibnitz rule**

If $\phi(\alpha) = \int_a^b f(x, \alpha)\, dx$ where a and b are constants, then

$$\phi'(\alpha) = \frac{d\phi}{d\alpha} = \int_a^b \frac{\partial}{\partial\alpha}[f(x, \alpha)]\, dx$$

Reduction formulae

➤ $$\int_0^{\pi/2} \sin^n x \, dx = \int_0^{\pi/2} \cos^n x \, dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \times k$$

where $k = 1$ when n is odd and $k = \pi/2$ when n is even.

➤ $$\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{[(m-1)(m-3)\ldots][(n-1)(n-3)\ldots]}{(m+n)(m+n-2)(m+n-4)\ldots} \times k$$

where $k = \pi/2$ only when m and n are even integers.

Applications of Integral calculus

Derivative of Arc Length

(i) $\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$ (ii) $\frac{ds}{dy} = \sqrt{1+\left(\frac{dx}{dy}\right)^2}$

(iii) $\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$ (iv) $\frac{ds}{d\theta} = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}$

(v) $\frac{ds}{dr} = \sqrt{1+\left(r\frac{d\theta}{dr}\right)^2}$

Integration with respect to the corresponding independent variable will give s.

Applications formulat at a glance

	Cartesian curve	Parametric curve	Polar curve
Area (A)	$\int_a^b y\,dx$ or $\int_c^d x\,dy$	$\int_{t_1}^{t_2} y\frac{dx}{dt}dt$ or $\int_{t_1}^{t_2} x\frac{dy}{dt}dt$	$\frac{1}{2}\int_{\theta_1}^{\theta_2} r^2\,d\theta$
Length (s)	$\int_a^b \frac{ds}{dx}dx$ or $\int_c^d \frac{ds}{dy}dy$	$\int_{t_1}^{t_2} \frac{ds}{dt}dt$	$\int_{\theta_1}^{\theta_2} \frac{ds}{d\theta}d\theta$ or $\int_{r_1}^{r_2} \frac{ds}{dr}dr$